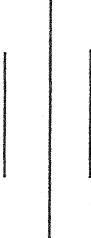
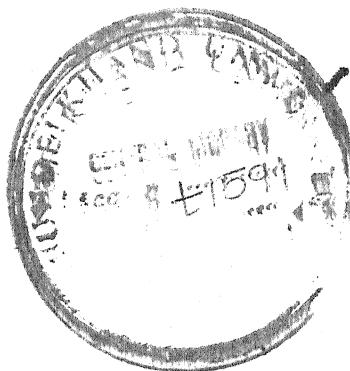


CERTAIN INVESTIGATIONS IN THE FINSLERIAN DIFFERENTIAL GEOMETRY

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CERTIFICATE

This is to certify that the thesis embodies the work of the candidate himself, that the candidate has worked under me with effect from 13th March 1995, and that the candidate has put in the required attendance during the said research period.

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PPMishra
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PREFACE

This thesis is an outcome of the research work carried out by the author under the supervision and guidance of Dr. H.S. Shukla, Principal, Pt. Jawahar Lal Nehru Post-Graduate College, BANDA (U.P.), INDIA.

The thesis has been divided into five chapters. Each chapter is subdivided into a number of articles. The references quoted in the thesis are in the form (c.a.e.), where c,a and e stand for the numbers of chapter, article and equation respectively. If c coincides with the chapter at hand, it has been omitted. The figures in the square brackets refer to the reference given at the end of the chapter. The notations ∂_k and $\dot{\partial}_k$ denote the partial derivatives with respect to x^k and y^k respectively.

The first chapter is introductory and deals with the basic definitions of Finslerian geometry. It contains important formulae frequently used in the subsequent chapters.

In the second chapter we have considered an n-dimensional affinely connected Finsler space ([89]) equipped with $2n$ -line elements (x^i, \dot{x}^i) and the fundamental metric function $F(x, \dot{x})$ which is positively homogeneous of degree one in its directional arguments. If the projective deviation tensor field $W^i_{hjk}(x, \dot{x})$ satisfies the condition: $W^i_{hjk}(s) = \lambda_s W^i_{hjk}$, where $\lambda_s(x)$ is a non-zero covariant vector field, the space has been called a WRF_n space. In such a space the infinitesimal point transformation $\bar{x}^i = x^i + v^i(x) d\tau$, where $v^i(x)$ is a contravariant vector field and $d\tau$ is an infinitesimal constant, is called a projective affine motion if and only if $L_v G^i_{jk} = 0$, where L_v denotes the Lie – derivative with respect to the

said point transformation. We have studied a WRF_n space admitting the said infinitesimal transformation and satisfying $L_v \lambda_s = 0$. For brevity, we have called such a restricted space an $SWRF_n$ space. We have discussed the properties of $SWRF_n$ space in detail. We have also obtained the necessary and sufficient condition for the existence of projective affine motion in a W – recurrent Finsler space.

In 1966, M. Matsumoto ([53]) determined uniquely Cartan's connection by assuming the four axioms : (1)The connection is metrical, (2) The deflection tensor field vanishes, (3) The torsion tensor field T vanishes and (4) The torsion tensor field S Vanishes. In 1969, M. Hashiguchi ([27]) replaced the condition (2) by some weaker condition and determined the Finsler connection with the given deflection tensor field. In 1975, ([26]) he also determined uniquely a Finsler connection by replacing the condition (3). In almost all these works, it has been assumed that the connection is metrical. In 1990, Prasad, Shukla and Singh ([85]) have introduced a Finsler connection with respect to which the metric tensor is h -recurrent. In the said introduction they have assumed that the v -covariant derivative of metric tensor vanishes and the torsion tensor fields T and S also vanish.

The purpose of the third chapter is to introduce a Finsler connection, which is neither h -metrical nor v -metrical, but with respect to which both the h -and v -covariant derivatives of metric tensor are recurrent. Such a Finsler connection will be called hv -recurrent Finsler connection. We have discussed the curvature properties of hv -recurrent Finsler connection . As particular cases of this connection we have discussed h -recurrent Finsler connection and v -recurrent Finsler connection and their curvature properties.

Izumi ([45]), while studying the conformal transformation of Finsler spaces introduced the h-vector X_i which is v-covariantly constant with respect to Cartan's connection and satisfies the relation $LC_{ij}^h X_h = \rho h_{ij}$. Thus the h-vector X_i is not only a function of coordinates, but it is also a function of directional arguments satisfying $L \partial_j X_i = \rho h_{ij}$. Various transformations of the Finsler metric have been studied in the literature ([57]), ([85]) and ([86]). The purpose of the fourth chapter is to obtain the relation between the v-curvature tensors with respect to Cartan's connection of the Finsler spaces (M^n, L) and (M^n, L^*) , where $L^*(x, y)$ is obtained from $L(x, y)$ by the transformation $L^*(x, y) = L^2(x, y) + (X_i(x, y) y^i)^2$, where $X_i(x, y)$ is an h-vector in (M^n, L) . We have also studied the h-vector fields in (M^{n-1}, L) , where (M^{n-1}, L) is a hyper surface of (M^n, L) .

A Randers space is a Finsler space with metric $ds = \alpha + \beta$, where $\alpha = (g_{ij}(x) dx^i dx^j)^{1/2}$ is a Riemannian metric and $\beta = b_i(x) dx^i$ is a 1-form, which was introduced by G. Randers ([92]). In 1978, S. Numata ([80]) introduced a Finsler space with metric $ds = \mu + \beta$, where $\mu = (g_{ij}(x) dx^i dx^j)^{1/2}$ is a Minkowskian metric and β is a 1-form as stated above. Numata found the torsion tensors R_{hjk} and P_{hjk} of such a space.

In the fifth chapter, we have introduced a Finsler space with metric $ds = \mu + \beta$, where $\mu = (g_{ij}(y) y^i y^j)^{1/2}$ is a Minkowskian metric and $\beta = X_i(x, y) y^i$, where X_i is an h-vector in (M^n, L) . We have found out the torsion tensor R^*_{hjk} of $F^{*n} = (M^n, L^*)$, where $L^*(x, y)$ is obtained from $L(y)$ by the transformation: $L^*(x, y) = L(y) + \beta(x, y)$ and considered the case that this space be a space of scalar curvature. We have also determined the torsion tensor P^*_{hjk} and considered the case that this space be a Landsberg space.

A selected bibliography consisting of the references of a number of books and papers on the subject has been given in the end.

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CHAPTER - I

INTRODUCTION

1. COORDINATE, CURVE AND LINE ELEMENT

Let R be a region of an n -dimensional space X_n which is covered completely by a coordinate system, such that any point of R is represented by a set of n - independent variables x^i ($i = 1, 2, 3, \dots, n$), which are called the coordinates of the point.

A set of points of R , whose coordinates are expressed as functions of single parameter t , is regarded as a curve of X_n . Thus the equations:

$$(1.1) \quad x^i = x^i(t)$$

define a curve C of X_n . The vector y^i with the components

$$(1.2) \quad y^i = \frac{dx^i}{dt}$$

is called the tangent vector to C .

The combination (x^i, y^i) of $2n$ elements is called the line element of the curve C .

We shall restrict our attention to the region R , and when we refer to X_n or F^n in existing literature, it is to be understood that the restriction is implied.

2. FINSLER SPACE

Let a function $L(x^i, y^i)$ be defined for all the line elements of R and be of Class C^5 in all the $2n$ arguments. Suppose further that this function defines the distance ds between two points $P(x^i)$ and $Q(x^i + dx^i)$ of R by the relation :

$$(2.1) \quad ds = L(x^i, dx^i)$$

DEFINITION (2.1). The space X_n equipped with the fundamental function defining the metric (2.1) is called a Finsler space ([14], P. 3-5) provided that $L(x^i, y^i)$ satisfies the following conditions :-

CONDITION (A) : The function $L(x^i, y^i)$, is positively homogeneous of degree 1 in y^i , that is,

$$L(x^i, ky^i) = kL(x^i, y^i), \text{ for } k > 0.$$

CONDITION (B) : The function $L(x^i, y^i)$ is positive if not all y^i vanish simultaneously, i.e.,

$$L(x^i, ky^i) > 0, \text{ with } \sum_{i=1}^n (y^i)^2 \neq 0.$$

CONDITION (C) : The quadratic form

$[\partial^2 L^2(x^i, y^i) / \partial y^i \partial y^j] \xi^i \xi^j$ is positive definite for all variables ξ^i .

From Euler's theorem on homogeneous functions and condition(A), it follows that

$$(2.2) \quad [\partial L(x^i, y^i)/\partial y^i] y^i = L(x^i, y^i) \text{ and}$$

$$(2.3) \quad [\partial^2 L(x^i, y^i)/\partial y^i \partial y^j] y^j = 0$$

writing

$$(2.4) \quad g_{ij}(x^i, y^i) = \frac{1}{2} \partial^2 L^2(x^i, y^i)/\partial y^i \partial y^j,$$

we can deduce (from the theory of quadratic form and condition (C))

$$(2.5) \quad g(x^i, y^i) = |g_{ij}(x^i, y^i)| > 0$$

for all line elements (x^i, y^i) .

Suppose, in particular, the function L is of the form

$$(2.6) \quad L(x^i, dx^i) = [g_{ij}(x^i) dx^i dx^j]^{1/2},$$

where the coefficients $g_{ij}(x^i)$ are independent of dx^i .

The metric defined by such a function L is called a Riemannian metric and the space X_n is called a Riemannian space. The n -dimensional Finsler space will be denoted by F^n .

3. METRIC TENSOR

It can be verified easily that the set of quantities

$g_{ij}(x^i, y^i)$ defined by (2.4) form the components of a covariant tensor of order two. It is clear that $g_{ij}(x^i, y^i)$ are positively homogeneous of degree zero in y^i and are symmetric in covariant indices i and j .

Due to homogeneity condition (A) for the function $L(x^i, y^i)$, we have,

$$(3.1) \quad L^2(x, y) = g_{ij}(x, y) y^i y^j$$

where in the following we denote

$$x = (x^1, x^2, x^3, \dots, x^n) \text{ and } y = (y^1, y^2, y^3, \dots, y^n)$$

Since the rank of the matrix $\|g_{ij}\|$ is n , we have the inverse matrix $\|g^{ij}\|$, such that

$$(3.2) \quad g_{ij} g^{jk} = \delta_i^k,$$

where δ_i^k is Kronecker delta.

DEFINITION (3.1) : The tensor with covariant components g_{ij} and contravariant components g^{ij} is called the metric tensor or the first fundamental tensor of the Finsler space F^n .

We define the quantities

$$(3.3) \quad C_{ijk}(x, y) = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k} = \frac{1}{4} \frac{\partial^3 L^2}{\partial y^i \partial y^j \partial y^k}$$

which is positively homogeneous of degree -1 in y^i and symmetric in all its indices. These quantities named as Cartan's C-tensor satisfy the conditions.

$$(3.4) \quad C_{ijk} y^i = C_{ijk} y^j = C_{ijk} y^k = 0$$

and

$$(3.5) \quad \partial C_{ijk} / \partial y^h = \partial C_{ijh} / \partial y^k = \partial C_{ihk} / \partial y^j = \partial C_{hjk} / \partial y^i$$

4. TANGENT SPACE, MINKOWSKIAN SPACE, INDICATRIX

DEFINITION (4.1) : At a point $P(x^i)$ of X_n , the n -dimensional linear vector space whose elements are the quantities dx^i , obeying the law of transformation

$$(4.1) \quad dx^{i'} = (\partial x^{i'}/\partial x^j) dx^j$$

is called a tangent space at the point $P(x^i)$.

Every point of F^n can in this way be associated with a tangent space which is denoted by $T_n(P)$ or $T_n(x^i)$ ([16]).

DEFINITION (4.2) : A Finsler space F^n is called a Minknowkian space if there exists a coordinate system in which the metric function L is independent of x^i ([14]), p.50).

DEFINITION (4.3) : The indicatrix of F^n at a point x^i is defined by the equation $L(x, y) = 1$, (x^i is fixed), ([9], [14], p. 12)

DEFINITION (4.4) : A tensor T of F^n is called indicatory ([11]), if its components $T_{ij} \dots_k$ satisfy

$$T_{ij} \dots_k = T_{io} \dots_k = T_{ij} \dots_o = 0,$$

where and throughout the thesis 'o' denotes the contraction with y^i .

5. DUAL TANGENT SPACE

Corresponding to each directional argument y^i of $T_n(p)$ there exists a covariant vector y_i defined by

$$(5.1) \quad y_i = g_{ij}(x, y) y^j.$$

where it may be noted that the direction argument in g_{ij} coincide with the vector y^i under consideration .

Thus (5.1) assigns a set of the values y_i to each point y^i of $T_n(P)$. The quantities y_i may be regarded as the positional coordinates of points in a second space $T_n'(P)$. We call this space dual tangent space of X_n at P . So the dual tangent space $T_n'(P)$ is the totality of all covariant vectors attached to X_n at P .

The metric function of $T_n'(P)$ is a function L (x^i, y_i) satisfying the conditions corresponding to (A), (B) and (C) given in section 2.

6. MAGNITUDE OF A VECTOR. THE NOTION OF ORTHOGONALITY

The metric tensor $g_{ij}(x, y)$ may be used in two different ways, in defining the magnitude of a vector and also the angle between the two vectors.

DEFINITION (6.1) : Let v^i be a vector , then the scalar $|v|$ given by

$$(6.1) \quad |v|^2 = g_{ij}(x, v) v^i v^j$$

is called the magnitude of the vector v^i .

If w^i is another vector then the ratio

$$(6.2) \quad \cos(v, w) = [g_{ij}(x, v) v^i w^j] / L(x^i, v^i) L(x^i, w^i)$$

is called 'MINKOWSKIAN COSINE' corresponding to the ordered pair of directions v^i, w^i ([12]).

It is obvious from (6.2) that Minkowskian cosine is non - symmetric in v^i and w^i .

DEFINITION (6.2) : Let v^i be a vector and y^i be an arbitrarily fixed direction, then the scalar v , defined by

$$(6.3) \quad v^2 = g_{ij}(x, y) v^i v^j$$

is said to be the magnitude of the vector, v^i corresponding to pre assigned direction y^i .

If w^i is another vector , the ratio

$$(6.4) \quad \cos(v, w) = [g_{ij}(x, y) v^i w^j] / [g_{ij}(x, y) v^i v^j]^{1/2} [g_{ij}(x, y) w^i w^j]^{1/2}$$

is called the cosine between the vectors v^i and w^i , for the direction y^i . This cosine is symmetric in v^i and w^i .

To distinguish between the two magnitudes we call the magnitude given by (6.1) as the Minkowskian magnitude of v^i and that given by (6.3) the magnitude of v^i .

DEFINITION (6.3) : The vector w^i is said to be orthogonal with respect to v^i if

(6.5) $g_{ij}(x, v) v^i w^j = 0$. Thus, according to this definition if w^i is orthogonal with respect to v^i then it is not necessary that v^i is also orthogonal with respect to w^i .

DEFINITION(6.4): The vectors v^i and w^i are called orthogonal(for a pre-assigned direction y^i) if

(6.6) $g_{ij}(x, y) v^i w^j = 0$. This definition of orthogonality is symmetric in v^i and w^i .

7. CONNECTIONS AND COVARIANT DIFFERENTIATION IN F^n

FINSLER CONNECTION : The Finsler connection $F\Gamma$ is a triad $(F_{jk}^i, N_k^i, C_{jk}^i)$ of a v - connection F_{jk}^i , a non-linear connection N_k^i and a vertical connection C_{jk}^i ([7], 11]). In general, the vertical connection C_{jk}^i is different from Cartan's C - tensor obtained from C_{ijk} given by (3.3). However, there are certain Finsler connections to be discussed later on, in which these two quantities are identical.

Let T_j^i be a tensor field of $(1,1)$ type. The h - and v - covariant derivatives of T_j^i are defined by

$$(7.1) \quad T_{jk}^i = \delta T_j^i / \delta x^k + T_m^m F_{mk}^i - T_m^i F_{jk}^m$$

and

$$(7.2) \quad T_{jk}^i = \partial T_j^i / \partial y^k + T_m^m C_{mk}^i - T_m^i C_{jk}^m$$

respectively, where

$$(7.3) \quad \delta / \delta x^k = \partial / \partial x^k N^m_k \partial / \partial y^m$$

Various Finsler connections may be defined with the help of a Finsler metric. Some of the well known examples of Finsler connections in F^n are given below.

(A) RUND CONNECTION : The Christoffel's symbols of first and second kinds have been defined as ([14]), p.51)

$$(7.4) \quad \gamma_{hij}(x, y) = \frac{1}{2} [\partial g_{hi} / \partial x^j + \partial g_{ij} / \partial x^h - \partial g_{jh} / \partial x^i]$$

$$(7.5) \quad \gamma_{ij}^h = g^{hk}(x, y) \gamma_{hik}(x, y)$$

We define further

$$(7.6) \quad \Gamma_{ij}^h(x, y) = \gamma_{ij}^h(x, y) - C_{im}^h(x, y) \gamma_{mj}^m(x, y) y^p,$$

where

$$(7.7) \quad C_{ij}^h \stackrel{\text{def.}}{=} g^{hk}(x, y) C_{ikj}(x, y)$$

and C_{ikj} is defined by (3.3)

For a vector V^i , the components $\delta V^i / \delta t$ defined by

$$(7.8) \quad \delta V^i / \delta t = dV^i / dt + \Gamma_{jk}^i (x, y) V^j dx^k / dt$$

form the contravariant components of a vector.

The process of differentiation given by (7.8) is called δ - differentiation. This differentiation gives rise to a well defined parallel displacement. The vector $V^i + dV^i$ of $T_n (x^i + dx^i)$ is said to result from V^i of $T_n (x^i)$ by parallel displacement if $\delta V^i = 0$. Hence, for such a displacement, we have ([14]), p.55)

$$(7.9) \quad dV^i = - \Gamma_{jk}^i (x, y) V^j dx^k$$

The δ - derivative with respect to x^k (in the direction y^j) of an arbitrary tensor $T_j^i (x, \xi)$, is given by the formula ([14]), p.60)

$$(7.10) \quad T_{jk}^i = \partial T_j^i / \partial x^k + (\partial T_j^i / \partial y^h) \partial \xi^h / \partial x^k + T_m^m \Gamma_{mk}^{* i} (x, y) - T_m^i \Gamma_{mk}^{* m} (x, y) .$$

$$\text{where } \Gamma_{jk}^{* i} (x, y) = g^{im} (x, y) \Gamma_{mk}^{* i} (x, y)$$

and

$$(7.11) \quad \Gamma_{mk}^{* i} = \gamma_{mk} (x, y) - \{ C_{kmh} (x, y) \Gamma_{ji}^h (x, y)$$

$$+ C_{mjh}(x,y) \Gamma^h_{ki}(x,y) - C_{jkh}(x,y) \Gamma^h_{mi}(x,y) \} y^i.$$

It may be verified that γ^i_{jk} and $\Gamma^*{}^i_{jk}$ are symmetric in its lower indices j and k , while Γ^i_{jk} is non-symmetric in j and k . Also we have

$$(7.12) \quad \Gamma^*{}^i_{jk} y^j y^k = \Gamma^i_{jk} y^j y^k = \gamma^i_{jk} y^j y^k$$

$$(7.13) \quad \Gamma^i_{jk} y^k = \Gamma^*{}^i_{jk} y^k$$

and

$$(7.14) \quad \Gamma^i_{jk} y^j = \gamma^i_{jk} y^j$$

The partial δ - derivative of metric tensor g_{ij} does not vanish in general. This is a significant observation as it leads to a development of the geometry of Finsler spaces which differs considerably from the Riemannian geometry in which the covariant derivative of the metric tensor is zero. Further, it is to be noted that if the vector field ξ^i is stationary, i.e., $\xi^i_{,j} = 0$ then the partial δ - differentiation of a tensor field is h - covariant derivative with respect to Rund connection ($\Gamma^*{}^i_{jk}$, G^i_k , 0), where $\Gamma^*{}^i_{jk}$ is V - connection defined by (7.11) and G^i_k is defined by

$$(7.15) \quad G^i_k(x, y) = \partial G^i / \partial y^k, \quad 2G^i(x, y) = \gamma^i_{jk} y^j y^k$$

The vertical connection vanishes in case of Rund's Finsler connection. Hence the ∇ - covariant derivative of a tensor field is identical to the partial derivative with respect to the element of support y^i ([5], [11]).

(B)CARTAN CONNECTION : E. Cartan ([2]) introduced a system of axioms to give uniquely a Finsler connection from the fundamental function $L(x, y)$. However, according to M. Matsumoto ([10], [11]) Cartan's axioms are equivalent to the following :

$$(7.16) \quad \begin{array}{ll} (a) \quad g_{ijk} = 0, & (b) \quad g_{ij}|_k = 0, \\ (c) \quad F_{jk}^i - F_{kj}^i = 0, & (d) \quad C_{jk}^i - C_{kj}^i = 0, \\ (e) \quad y^h F_{hj}^i - N_j^i = 0. & \end{array}$$

The Cartan connection of F^n is denoted by

$C\Gamma = (\Gamma^*{}^i_{jk}, \Gamma^*{}^i_{oj}, C^i_{jk})$. The axioms (7.16 b) and (7.16d), in view of (7.2), give

$$(7.17) \quad C^i_{jk} = \frac{1}{2} g^{ih} \partial g_{ik} / \partial y^h$$

This shows that the vertical connection and Cartan's C-tensor are identical. Further, axioms (7.16 a) and (7.16 c) in view of relation (7.17) and (7.1) yield

$$(7.18) \quad F_{ijk} = F_{ik}^h g_{jh} = \gamma_{ijk} - C_{ijm} N^m_k - C_{jkm} N^m_i + C_{kim} N^m_j$$

Contracting the equation (7.18) with

$g^{ih} y^j$ and applying axiom (7.16e), we get

$$(7.19) \quad N^h_k = \gamma^h_{ik} y^j - C^h_{km} N^m_j y^i.$$

Again contraction of (7.19) with y^k , gives

$$(7.20) \quad N^h_k y^k = \gamma^h_{kk} y^i y^k$$

Substituting (7.19) and (7.20) in (7.18), we get

$$F_{ijk} = \Gamma^*_{ijk},$$

where Γ^*_{ijk} is defined by (7.11). Thus the Cartan's v-connection are identical with the Rund's v-connection and it is given by (7.11). The Cartan's non-linear connection N^i_j is obtained from (7.19) after substituting from (7.20).

$$(7.21) \quad N^i_j = \gamma^i_{kj} y^k - C^i_{jm} \gamma^m_{hl} y^h y^l = G^i_j = \Gamma^*_{oj}.$$

The Cartan's vertical connection C_{jk}^i is given by (7.17). It is easy to verify from the axioms (7.16a), (7.16c) and equation (3.1) that

$$(7.22) \quad (a) \quad y^i_{|h} = 0, \quad (b) \quad L_{|h} = 0, \quad (c) \quad l^i_{|h} = 0;$$

where l^i is a unit vector in the direction of the element of support y^i , i.e.,

$$l^i = y^i/L(x, y)$$

Since C_{jk}^i is an indicatory tensor, therefore from (7.2), we have

$$y^i_{|h} = \delta^i_h$$

This in view of (3.1) and axiom (7.16a), gives,

$$(7.23) \quad L_i = L_{|i} = \partial L / \partial y^i = l_i,$$

$$\text{where } l_i = g_{ij} l^j$$

It may also be verified that

$$(7.24) \quad (a) \quad l^i_{|j} = L^{-1} h^i_j, \quad (b) \quad l_{ij} = 0,$$

$$(c) \quad l_{ij} = L^{-1} h_{ij}$$

$$(7.25) \quad (a) \quad h_{ijk} = 0, \quad (b) \quad h_{ijk}^l = -L^{-1} (l_i h_{jk} + l_j h_{ki}),$$

where h_{ij} are components of angular metric tensor

defined by

$$(7.26) \quad h_{ij} = g_{ij} - l_i l_j$$

$$\text{and} \quad h_j^k = g^{ik} h_{jk}$$

(C) BERWALD CONNECTION : Berwald ([1]) introduced a connection parameter G_{jk}^i given by

$$(7.27) \quad G_{jk}^i (x, y) = \partial^2 G^i / \partial y^j \partial y^k,$$

$$\text{where} \quad 2G^i (x, y) = \gamma_{jk}^i (x, y) y^j y^k.$$

According to him the covariant derivative of any tensor T_j^i is defined by the relation

$$T_{j(k)}^i = \delta T_j^i / \delta x^k + T_j^m G_{mk}^i - T_m^i G_{jk}^m,$$

$$\text{where} \quad \delta / \delta x^k = \partial / \partial x^k - G^m_k \partial / \partial y^m$$

and G^m_k is given by (7.15).

Thus Berwald's v- connection and non- linear connections are G_{jk}^i and G_j^i respectively. The vertical connection vanishes in case of Berwald connection ([5], [10]).

The relation between Berwald's and Cartan's v- connections G_{jk}^i and Γ_{jk}^{*i} is ([14], p. 79-81),

$$(7.28) \quad G_{jk}^i = \Gamma_{jk}^{*i} + P_{jk}^i, \text{ where,}$$

$$(7.29) \quad P^i_{jk}(x,y) = C^i_{jk|0} = [\partial \Gamma^*{}^i_{jp} / \partial y^k] y^p \\ = [\partial \Gamma^*{}^i_{kp} / \partial y^j] y^p.$$

Further, the Berwald's h - covariant derivative of metric tensor g_{ij} is given by ([14], p.80)

$$(7.30) \quad g_{ij(k)} = -2 P_{ijk}, \text{ where}$$

$$(7.31) \quad P_{ijk} = g_{jh} P^h_{ik} = C_{ijk|0}$$

It may be noted that the tensor P_{ijk} is a symmetric and indicatory tensor. Also we have the relations:

$$(7.32) \quad L_{(i)} = 0, I^i_{(j)} = 0$$

$$I_{(i(j))} = 0, h^i_{j(k)} = 0, h_{ij(k)} = -2 P_{ijk}.$$

8. CURVATURE TENSORS IN F^n

Several curvature tensors have been defined and studied in Finsler spaces with the help of different Finsler connections. We introduce some of them.

The Rund's curvature tensor K^i_{jhk} is defined as

([14], p.97)

$$(8.1) \quad K^i_{jhk} = \mathcal{L}_{(h, k)} (\delta \Gamma^*{}^i_{jh} / \delta x^k + \Gamma^*{}^m_{jh} \Gamma^*{}^i_{mk}),$$

where and throughout the thesis $\mathcal{C}_{(h, k)}$ (-----) denotes the interchange of the indices h, k and subtraction. This curvature tensor and its corresponding theory has not been used in the present work. We therefore leave the further discussion of this curvature tensor.

The Ricci identities for a tensor T^i_j involving h - and v - covariant derivatives with respect to Cartan connection are given by ([8]).

$$(8.2) \quad T^i_{j|p|l} - T^i_{j||p} = T^h_j R^i_{hpl} - T^i_h R^h_{jpl} - T^i_{jh} R^h_{pl},$$

$$(8.3) \quad T^i_{j|p|l} - T^i_{j||p} = T^h_j P^i_{hpl} - T^i_h P^h_{jpl} - T^i_{jh} C^h_{pl} - T^i_{jh} P^h_{pl},$$

$$(8.4) \quad T^i_{j|p|l} - T^i_{j||p} = T^h_j S^i_{hpl} - T^i_h S^h_{jpl},$$

where

$$(8.5) \quad R^i_{hkl} = \mathcal{C}_{(k, l)} \{ \delta \Gamma^{*i}_{hk} / \delta x^l + \Gamma^{*m}_{hk} \Gamma^{*i}_{ml} \}$$

$$+ C^i_{hm} R^m_{kl},$$

$$(8.6) \quad R^i_{kl} = R^i_{hkl} y^h = \delta G^i_k / \delta x^l - \delta G^i_l / \delta x^k$$

$$(8.7) \quad P^i_{hkl} = \partial \Gamma^{*i}_{hk} / \partial y^l - C^i_{hlk} + C^i_{hm} P^m_{kl}$$

and

$$(8.8) \quad S^i_{hkl} = C^m_{hl} C^i_{mk} - C^m_{hk} C^i_{ml}$$

The tensors defined by (8.5), (8.7) and (8.8) are called Cartan's curvature tensors and the tensors defined by (7.17), (7.29) and (8.6) are called torsion tensors. The nomenclature of these tensors is summarized as follows:

R^i_{hkl} ----- h - curvature tensor,

P^i_{hkl} ----- hv - curvature tensor,

S^i_{hkl} ----- v - curvature tensor,

R^i_{kl} ----- (v) h - torsion tensor,

P^i_{kl} ----- (v) hv - torsion tensor,

C^i_{hl} ----- (h) hv - torsion tensor.

It is to be remarked here that S^i_{hkl} is the Riemann's curvature tensor of the tangent Riemannian space at any point of F^n . The tensors S^i_{jkl} and P^i_{jk} are indicatory tensors. The curvature and torsion tensors satisfy the following identities.

$$(8.9) \quad (a) \quad R_{hikl} = - R_{hilk},$$

$$(b) \quad R_{hikl} = - R_{ihkl},$$

$$(8.10) \quad (a) \quad P_{hijk} = - P_{ihjk},$$

$$(b) \quad P_{hikj} = - P_{hijk} = - S_{hikj|0},$$

$$(c) \quad P_{hikj} y^h = P_{ikj} = C_{ikj|0}$$

$$(8.11) \quad (a) \quad S_{hikj} = - S_{hijk},$$

$$(b) \quad S_{hikj} = - S_{ihkj},$$

$$(c) \quad S_{hikj} = - S_{kjhi},$$

$$(8.12) \quad R_{ikj} = - R_{ijk},$$

$$(8.13) \quad P_{ijk} = P_{ikj} = P_{jik} = P_{kji}$$

where

$$R_{hikj} = g_{mi} R^m{}_{hjk}, \quad P_{hikj} = g_{im} P^m{}_{hkj},$$

$$S_{hipj} = g_{im} S^m{}_{hpj},$$

$$R_{ipj} = R_{hipj} y^h = g_{mi} R^m{}_{pj}$$

and

$$P_{ipj} = P^m{}_{pj} g_{mi}$$

The expression for hv - curvature tensor P_{ijkl} is given by

$$(8.14) \quad P_{ijkl} = \mathcal{L}_{(i,j)} (C_{jkl|i} + C_{ikh} P^h{}_{jl})$$

This curvature tensor can also be written in the form ([11])

$$(8.15) \quad P_{hijk} = \mathcal{L}_{(h,i)} (P_{ijk|h} + C_{ikl} P^l{}_{jh})$$

which gives

$$(8.16) \quad P_{hijo} = 0 = P_{hiok}.$$

9. HYPERSURFACE.

The theory of subspaces of a Finsler space has been studied by Davies ([3]), Hombu ([6]), Rund ([13]), Eliopoulos ([4]) and several other authors. The present section is devoted in outlining the properties of the manifolds immersed in a Finsler space equipped with a general Finsler connection $F\Gamma$.

DEFINITION (9.1). The totality of all the points of F^n whose coordinates x^i can be expressed as a function of m parameters ($m < n$) is called a subspace F^m of dimension m of the space F^n .

A subspace may be represented parametrically by the equations

$$x^i = x^i(u^\alpha), \quad [(i = 1, 2, 3, \dots, n), \quad (\alpha = 1, 2, 3, \dots, m)]$$

Subject to the condition that the rank of the matrix $\| \partial x^i / \partial u^\alpha \|$ is m .

An $(n-1)$ dimensional subspace of F^n is called a hypersurface. Since we have studied the properties of Finsler hypersurface in this thesis, therefore, some basic formulas of a hypersurface immersed in an n -dimensional Finsler space F^n are given below.

The projection factor B_α^i defined by

$$(9.2) \quad B_\alpha^i(u) = \partial x^i / \partial u^\alpha$$

behaves like contravariant vector in F^n . Along the coordinate curve of the parameter u^α of F^{n-1} , the vector B_α^i (α fixed, $i = 1, 2, \dots, n$) is tangent to the curve. Thus B_α^i ($\alpha = 1, 2, \dots, n-1$, $i = 1, 2, \dots, n$) may be regarded as $(n-1)$ linearly independent vectors tangent to F^{n-1} at its any point and a vector x^i tangent to F^{n-1} at the point may be expressed uniquely in the form,

$$x^i = B_\alpha^i X^\alpha ,$$

Where X^α are components of the vector with respect to the coordinate system (u^α) .

To introduce a Finsler structure in F^{n-1} , the supporting element y^i at a point (u^α) of F^{n-1} is assumed to be tangential to F^{n-1} , so that we may write

$$(9.3) \quad y^i = B^i_\alpha v^\alpha$$

Thus v^α is thought of as the supporting element of F^{n-1} at the point (u^α) .

Denoting y^i of (9.3) by $y^i(u, v)$, the induced metric function of F^{n-1} is given by

$$(9.4) \quad \underline{L}(u, v) = L\{x(u), y(u, v)\}.$$

In the following we shall use the notations

$$\begin{aligned} B^i_{\alpha\beta} &= \partial B^i_\alpha / \partial u^\beta, \quad B^i_{\alpha\beta} = B^i_{\alpha\beta} v^\alpha, \quad B^{ij..k}_{\alpha\beta..\gamma} \\ &= B^i_\alpha B^j_\beta \dots B^k_\gamma \end{aligned}$$

The induced metric $\underline{L}(u, v)$ yields $l_\alpha = \partial \underline{L} / \partial v^\alpha$,
the metric tensor

$g_{\alpha\beta} = 1/2 \partial^2 L^2 / \partial v^\alpha \partial v^\beta$ and Cartan's C-tensor

$$C_{\alpha\beta\gamma} = 1/2 \partial g_{\alpha\beta} / \partial v^\gamma \text{ of } F^{n-1}$$

From (9.4), it is easy to show that

$$(9.5) \quad l_\alpha = l_i B_\alpha^i ,$$

$$(9.6) \quad g_{\alpha\beta} = g_{ij} B_{\alpha\beta}^{ij}$$

$$(9.7) \quad C_{\alpha\beta\gamma} = C_{ijk} B_{\alpha\beta\gamma}^{ijk}$$

At each point (u^α) of F^{n-1} , a unit normal vector

$N^i(u, v)$ is defined by

$$(9.8) \quad g_{ij} B_\alpha^i N^j = 0 ,$$

$$(9.9) \quad g_{ij} N^i N^j = 1 .$$

Making use of the inverse matrix $\|g^{\alpha\beta}\|$ of $\|g_{\alpha\beta}\|$,

the inverse projection factor B_α^i is defined by

$$(9.10) \quad B_\alpha^i(u, v) = g^{\alpha\beta}(u, v) g_{ij}(x, y) B_\beta^j .$$

The following identities are satisfied in F^{n-1} .

$$(9.11) \quad (a) \quad B_\alpha^i B_\beta^j = \delta_\beta^i - N^i N_j ,$$

$$(b) \quad B_\alpha^i B_\beta^j = \delta_\alpha^\beta ,$$

$$(c) \quad B_\alpha^i N_i = 0 ,$$

$$(d) \quad N^i B^\alpha_i = 0,$$

$$(e) \quad l_i N^i = l^i N_i = 0,$$

$$(f) \quad N_i = g_{ij} N^j,$$

$$(g) \quad g^{ij} N_i N_j = 1.$$

We also have

$$(9.12) \quad g^{\alpha\beta} (u, v) = g^{ij} (x, y) B^\alpha_i B^\beta_j,$$

which gives

$$(9.13) \quad g^{\alpha\beta} B^{ik}_{\alpha\beta} = g^{ik} - N^i N^k.$$

As for the angular metric tensor h_{ij} the relations

(7.26), (9.5), (9.6), (9.8), (9.9) and (9.11 e) yield

$$(9.14) \quad h_{\alpha\beta} = h_{ij} B^{ij}_{\alpha\beta},$$

$$(9.15) \quad (a) \quad h_{ij} N^i B^j_\alpha = 0,$$

$$(b) \quad h_{ij} N^i N^j = 1,$$

We shall use the following indicatory tensors defined from Cartan's C-tensor.

$$(9.16) \quad (a) \quad M_{ij} (x, y) = C_{ijk} N^k,$$

$$(b) \quad M_i(x, y) = M_{ij} N^j,$$

$$(9.17) \quad (a) \quad M_{\alpha\beta}(u, v) = M_{ij} B^{ij}_{\alpha\beta}$$

$$(b) \quad M_\alpha(u, v) = M_i B^i_\alpha.$$

We shall also use the quantities

$$(9.18) \quad (a) \quad M^i_j = g^{ih} M_{hj},$$

$$(b) \quad M^\alpha_\delta = g^{\alpha\beta} M_{\beta\delta}.$$

In the theory of hyper surface of Finsler space there are two connection parameters, namely intrinsic and induced, corresponding to which, we have two h – covariant derivatives of a tensor field. The intrinsic connection parameter $\Gamma^*{}^\alpha_{\beta\gamma}$ is defined with respect to the metric of F^{n-1} in a manner formally identical with the mode of definition of the coefficient $\Gamma^*{}^i_{jk}$ of F^n ([15]). The induced and intrinsic vertical connection of F^{n-1} are identical and denoted by $C^\alpha_{\beta\gamma}$ but the induced connection parameter $F^\alpha_{\beta\gamma}$ and intrinsic connection parameter $\Gamma^*{}^\alpha_{\beta\gamma}$ are not identical in general.

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CHAPTER II

PROJECTIVE AFFINE MOTION IN A W-RECURRENT FINSLER SPACE

1. INTRODUCTION

We consider an n -dimensional affinely
connected Finsler space [1] equipped with $2n$ line-elements (x^i, \dot{x}^i) and a
fundamental metric function $F(x, \dot{x})$ positively homogeneous of degree one in
its directional arguments. The fundamental metric tensor $g_{ij}(x, \dot{x})$ $\stackrel{\text{def}}{=}$ $\frac{1}{2} \dot{\partial}_i$
 $\dot{\partial}_j F^2(x, \dot{x})$ of the space is symmetric in its indices i and j . Let $T^i_j(x, \dot{x})$ be any
tensor field depending on both the positional and directional arguments. The
covariant derivative of $T^i_j(x, \dot{x})$ with respect to x^k in the sense of Berwald is
given by

$$(1.1) \quad T^i_{j(k)} = \partial_k T^i_j - \dot{\partial}_h T^i_j G^h_k + T^h_j G^i_{hk} - T^i_h G^h_{jk},$$

$$(\partial_k = \partial/\partial x^k, \dot{\partial}_k = \partial/\partial \dot{x}^k)$$

where $G^i_{jk}(x, \dot{x})$ are Berwald's connection coefficients which
satisfy the following relations:

$$(1.2) \quad \dot{\partial}_h G^i_{jk} = G^i_{hjk}, \quad G^i_{hjk} \dot{x}^h = 0 \text{ and } G^i_{hk} = G^i_{kh}.$$

The commutation formulae involving the Berwald's covariant derivative are given by

$$(1.3) \quad \dot{\partial}_h T^i_{j(k)} - (\dot{\partial}_h T^i_j)_{(k)} = T^s_j G^i_{shk} - T^i_s G^s_{jhk}$$

$$(1.4) \quad T^i_{j(h)(k)} - T^i_{j(k)h} = -\partial_r T^i_j H^r_{hk} + T^s_j H^i_{shk} - T^i_s H^s_{jhk}$$

where

$$(1.5) \quad H^i_{hjk}(x, \dot{x}) \stackrel{\text{def}}{=} \partial_k G^i_{hj} - \partial_j G^i_{hk} + G^r_{hj} G^i_{rk} - G^r_{hk} G^i_{rj} \\ + G^i_{rhk} G^r_j - G^i_{rjh} G^r_k$$

is Berwald's curvature tensor field and satisfies the following relations:

$$(1.6) \quad H^i_{hjk} = -H^i_{hkj}, \quad H^i_{hjk} \dot{x}^h = H^i_{jk}, \quad H^r_{hij} = H_{hj} - H_{jh}, \\ H^r_{hjr} = H_{hj}, \quad H^i_i = (n-1)H, \quad \dot{x}^j H^i_{jk} = H^i_{jk}.$$

The projective deviation tensor field $W^i_{hjk}(x, \dot{x})$ of the space is given by

$$(1.7) \quad W^i_{hjk}(x, \dot{x}) = H^i_{hjk} + \frac{1}{(n+1)} \{ \delta^i_h H^r_{rjk} + \dot{x}^i \dot{\partial}_h H^r_{rjk} \} \\ + \frac{\delta^i_j}{(n^2-1)} (n H_{hk} + H_{kh} + \dot{x}^r \dot{\partial}_h H_{kr}) - \frac{\delta^i_k}{(n^2-1)} (n H_{hj} + H_{jh} + \dot{x}^r \dot{\partial}_h H_{jr})$$

and satisfies the following identity:

$$(1.8) \quad W^i_{hjk(l)} + W^i_{hkl(j)} + W^i_{hlj(k)} = 0.$$

satisfies the condition

$$(1.9) \quad W^i_{hjk(s)} = \lambda_s \ W^i_{hjk},$$

where $\lambda_s(x)$ is a non-zero covariant vector, the space is called a W-recurrent Finsler space or a WRF_n space.

Let us consider the infinitesimal point transformation

$$(1.10) \quad \bar{x}^i = x^i + v^i(x) d\tau,$$

where $v^i(x)$ is any vector field and $d\tau$ is an infinitesimal constant. The above transformation which is considered at each point in the space is called a projective affine motion when and only when

$$(1.11) \quad L_v G^i_{jk} = 0,$$

where L_v denotes the Lie-derivative with respect to (1.10).

The Lie-derivatives of the tensor field $T^i_j(x, \dot{x})$ and connection coefficient $G^i_{jk}(x, \dot{x})$ in view of (1.10) and the Berwald's covariant derivative are respectively given by [2] : These are

$$(1.12) \quad L_v T^i_j = T^i_{j(h)} v^h + T^i_h v^h_{(j)} - T^h_j v^i_{(h)} + \partial_h T^i_j v^h_{(s)} \dot{x}^s,$$

and

$$(1.13) \quad L_v G^i_{jk} = v^i_{(j)(k)} + H^i_{jk} v^h + G^i_{sjk} v_{(r)} \dot{x}^r$$

We have the following commutation formulae :

$$(1.14) \quad L_v (\partial_1 T^i_j) - \partial_1 L_v T^i_j = 0.$$

$$(1.15) \quad (L_v G^i_{jh})_{(k)} - (L_v G^i_{kh})_{(j)} = L_v H^i_{hjk} + \dot{x}^s G^i_{rjh} L_v G^r_{ks} - \dot{x}^s G^i_{rkh} L_v G^r_{js}$$

and

$$(1.16) \quad (L_v T^i_{jk(m)}) - (L_v T^i_{jk})_{(m)} = T^s_{jk} L_v G^i_{sm} - T^i_{sk} L_v G^s_{jm} - T^i_{js} L_v G^s_{km} - \partial_s T^i_{jk} L_v G^s_{rm} \dot{x}^r.$$

Hence, for an infinitesimal projective affine motion the last relation shows that the two operators L_v and (k) are commutative with each other.

With the help of the equations (1.11) and (1.15), we get

$$(1.17) \quad L_v H^i_{hjk} = 0$$

In view of the equations (1.6) and the fact that the operations of contraction and Lie-differentiation are commutative, the above relation yields:

$$(1.18) \quad L_v H^r_{ijk} = 0, \quad L_v H_{jk} = 0, \quad L_v H = 0.$$

Taking the Lie- derivative of the either side of (1.7) and using the equations(1.14), (1.17) and (1.18), we obtain

$$(1.19) \quad L_v W^i_{hjk} = 0$$

Applying L_v to the both sides of (1.9) and using the equations (1.11), (1.16) and (1.19), we have

$$(1.20) \quad (L_v \lambda_s) W^i_{hjk} = 0$$

Since the space is not an isotropic, we have

$$(1.21) \quad L_v \lambda_s = 0$$

i.e. the recurrence vector λ_s of the space must be Lie-invariant one.

In what follows, we shall study a WRF_n space admitting an infinitesimal transformation $\bar{x}^i = x_i + v^i (dx) dt$ which satisfies (1.21). For brevity we shall call such a restricted space a $SWRF_n$ space.

2. THE VANISHING OF $W^i_{hjk}(x, \dot{x})$

First of all we shall prove the following lemma:

LEMMA (2.1) : In an SWRF_n space if the recurrence vector is a gradient one, we have $\lambda_s v^s = \text{const.}$

PROOF : Let us put

$$(2.1) \quad \delta = \lambda_s v^s$$

Then with the help of the equations (1.12) and (1.21), we have

$$(2.2) \quad L_v \lambda_s = \lambda_{s(m)} v^m + \lambda_m v^m {}_{(s)} = 0$$

By virtue of the assumption $\lambda_{s(m)} = \lambda_{m(s)}$ the above equation reduces to

$$(2.3) \quad \delta {}_{(m)} = 0$$

which completes the proof.

In view of (1.12), the Lie-derivative of $W^i_{hjk}(x, \dot{x})$ is given by

$$(2.4) \quad L_v W^i_{hjk} = W^i_{hjk} v^s + W^i_{sjk} v^s {}_{(h)} + W^i_{hsk} v^s {}_{(j)} + W^i_{hjs} v^s {}_{(k)} - W^s {}_{hjk} v^i {}_{(s)} + \dot{\partial}_s W^i_{hjk} v^s {}_{(r)} \dot{x}^r.$$

which by virtue of the equation (1.9) and (2.1) reduces to

$$(2.5) \quad L_v W^i_{hjk} = \delta W^i_{hjk} + W^i_{sjk} v^s_{(h)} + W^i_{hsk} v^s_{(j)} + W^i_{hjs} v^s_{(k)} - W^s_{hjk} v^i_{(s)} + \partial_s W^i_{hjk} v^s_{(r)} \dot{x}^r.$$

Introducing the commutation formula (1.4) to the tensor field $W^i_{hjk}(x, \dot{x})$, we get

$$(2.6) \quad W^i_{hjk(l)(m)} - W^i_{hjk(m)(l)} = \partial_r W^i_{hjk} H^r_{slm} \dot{x}^s + W^s_{hjk} H^i_{slm} - W^i_{sjk} H^s_{hlm} - W^i_{hsk} H^s_{jlm} - W^i_{hjs} H^s_{klm}.$$

In view of the definition (1.9), the above relation reduces to

$$(2.7) \quad (\delta_{l(m)} - \delta_{m(l)}) W^i_{hjk} = -\partial_r W^i_{hjk} H^r_{slm} \dot{x}^s + W^s_{hjk} H^i_{slm} - W^i_{sjk} H^s_{hlm} - W^i_{hsk} H^s_{jlm} - W^i_{hjs} H^s_{klm}.$$

Next, let us assume that $\delta_m \neq \text{const}$. Then, with the help of the Lemma (2.1), we get

$$(2.8) \quad N_{lm}(x) \stackrel{\text{def}}{=} (\delta_{l(m)} - \delta_{m(l)}) \neq 0$$

Let us take

$$(2.9) \quad v^i_{(h)} = H^i_{hjk} q^{jk}$$

for a suitable non - symmetric tensor q^{jk} , then multiplying

(2.7) by q^{lm} and summing over l and m, we obtain

$$(2.10) \quad N_{lm} q^{lm} W^i_{hjk} = - \partial_r W^i_{hjk} v^r_{(s)} \dot{x}^s + W^s_{hjk} v^i_{(s)} - W^i_{sjk} v^s_{(h)} - W^i_{hsk} v^s_{(j)} - W^i_{hjs} v^s_{(k)}$$

Comparing the last equation with (2.5), we get

$$(2.11) \quad L_v W^i_{hjk} = (\delta - q^{lm} N_{lm}) W^i_{hjk}.$$

The above equation will vanish when and when

$$\delta \neq \text{const. and } N_{lm} \neq 0$$

From (2.5) and (2.7), we can construct the following identity:

$$(2.12) \quad N_{lm} L_v W^i_{hjk} = W^s_{hjk} (\delta H^i_{slm} - N_{lm} v^i_{(s)}) - W^i_{sjk} (\delta H^s_{hlm} - N_{lm} v^s_{(h)}) - v^s_{(h)} W^i_{hsk} (\delta H^s_{jlm} - N_{lm} v^s_{(j)}) - W^i_{hjs} (\delta H^s_{klm} - N_{lm} v^s_{(k)})$$

Thus, for $L_v W^i_{hjk} = 0$, the above equation yields ([6]):

$$(2.13) \quad \delta H^i_{slm} = N_{lm} v^i_{(s)}$$

where v^i does not mean a parallel vector field.

We put here the

DEFINITION 2.1 : An SWRF_n space satisfying $\lambda_m v^m \neq \text{const.}$ is called a special one of the first kind.

Next, let us again go back to the case,

$\lambda_m v^m = \text{const.}$ Of the foregoing lemma (2.1). Then, (2.7) is replaced by

$$(2.14) \quad -\dot{\partial}_r W^i_{hjk} H^r_{slm} \dot{x}^s + W^s_{hjk} H^i_{slm} - W^i_{sjk} H^s_{hlm} \\ - W^i_{hsk} H^s_{jlm} - W^i_{hjs} H^s_{klm} = 0$$

Transvecting it by q^{lm} and remembering the equation (2.9) we get

$$(2.15) \quad -\dot{\partial}_r W^i_{hjk} v^r_{(s)} \dot{x}^s + W^s_{hjk} v^i_{(s)} - W^i_{sjk} v^s_{(h)} - W^i_{hsk} v^s_{(j)} \\ - W^i_{hjs} v^s_{(k)} = 0$$

Substituting the above equation into the right hand side of (2.5), we obtain

$$(2.16) \quad L_v W^i_{hjk} = \delta W^i_{hjk}$$

Therefore when the arbitrary constant δ vanishes, we have

$$(2.17) \quad L_v W^i_{hjk} = 0$$

We put the

DEFINITION 2.2 : An SWRF_n space is called a special one of the second kind when

$$\lambda_m v^m = \text{const. holds good.}$$

Summarizing the above results, we have the following theorems.

THEOREM (2.1) : In a special SWRF_n space of the first kind, if the space has the resolved curvature H^i_{hjk} of the form (2.13),

$$L_v W^i_{hjk} = 0 \text{ holds good.}$$

THEOREM (2.2) : In a special SWRF_n space of the second kind, if the arbitrary constant $\lambda_m v^m$ vanishes, we have $L_v W^i_{hjk} = 0$.

From the last theorem, if $\lambda_m = 0$, then with the help of equation (1.9), we have

$$(2.18) \quad W^i_{hjk(r)} = 0.$$

Thus, we have

COROLLARY (2.1): In a symmetric Finsler space, $L_v W^i_{hjk} = 0$ is satisfied identically.

3. COMPLETE CONDITION

In this section we shall find the necessary and sufficient condition for (2.13). From the assumption (1.21), we have

$$(3.1) \quad L_v \lambda_m = \lambda_{m(s)} v^s + (\lambda_s v^s)_{(m)} - \lambda_{s(m)} v^s = 0.$$

By virtue of (2.1) and (2.8), the last equation reduces to

$$(3.2) \quad \delta_{(m)} + N_{ms} v^s = 0$$

In view of the equation (1.12), the Lie - derivative of $N_{lm}(x)$ is given by

$$(3.3) \quad L_v N_{lm} = N_{lm(s)} v^s + N_{sm} v^s_{(l)} + N_{ls} v^s_{(m)}$$

Remembering the commutation formula (1.16), we have

$$(3.4) \quad L_v (\lambda_{m(s)}) - (L_v \lambda_m)_{(s)} = - \lambda_r L_v G^r_{ms}$$

With the help of the equations (1.20), (1.21) and (2.8), the above relation reduces to

$$(3.5) \quad L_v N_{sm} = 0$$

Differentiating (2.7) covariantly with respect to x^n and using the equations (1.3), (1.9), (2.7) and (2.8) we obtain

$$(3.6) \quad N_{lm(n)} W^i_{hjk} = \lambda_n W^i_{hjk} N_{lm} + H^r_{alm} \dot{x}^a (W^s_{hjk} G^i_{srn} - W^i_{sjk} G^s_{hrn} - W^i_{hsk} G^s_{jrn} - W^i_{hjs} G^s_{krn})$$

Transvecting it by \dot{x}^n and noting the second equation of (1.2), we get after little simplifications:

$$(3.7) \quad N_{lm(n)} = \lambda_n N_{lm}$$

Thus, by virtue of the equations (3.3), (3.5) and (3.7), we get

$$(3.8) \quad \delta N_{lm} + N_{sm} v^s_{(l)} + N_{ls} v^s_{(m)} = 0.$$

Next, from the equation (3.2), we have

$$(3.9) \quad \delta_{(m)(n)} - \delta_{(n)(m)} = -(N_{ms} v^s)_{(n)} + (N_{ns} v^s)_{(m)}$$

δ being a non-constant scalar function, the above equation reduces to

$$(3.10) \quad N_{ms} v^s_{(n)} - N_{sn} v^s_{(m)} = -\lambda_n N_{ms} v^s + \lambda_m N_{ns} v^s$$

where, we have used (3.7) and $N_{ms} = -N_{sm}$. Substituting the last equation into the left hand side of (3.8), we get

$$(3.11) \quad \delta N_{mn} = -\lambda_n \delta_{(m)} + \lambda_m \delta_{(n)}$$

In an affinely connected space, the identity (1.8) reduces to

$$(3.12) \quad W^i_{hjk(l)} + W^i_{hkl(j)} + W^i_{hlij(k)} = 0$$

which in view of the definition (1.9) reduces to

$$(3.13) \quad \delta W^i_{hjk} = \lambda_k W^i_{hjl} v^l - \lambda_j W^i_{hks} v^s$$

where, we have used (2.1) and $W^i_{hjk} = -W^i_{hkj}$. Hence, from

(3.11) and (3.13), we can deduce the following identity:

$$(3.14) \quad \delta(\delta W^i_{hjk} - N_{jk} v^i_{(h)}) = \lambda_k (\delta W^i_{hjs} v^s + \delta_{(j)} v^i_{(h)}) - \lambda_j (\delta W^i_{hks} v^s + \delta_{(k)} v^i_{(h)})$$

Consequently (2.13) follows when and only

when, we have

$$(3.15) \quad \delta W^i_{hjs} v^s + \delta_{(j)} v^i_{(h)} = \lambda_j Q^i_h$$

where Q^i_h means a suitable tensor. Transvecting the above equation by v^j and summing over j , by virtue of $W^i_{hjk} v^j v^k = 0$ and $\delta_{(j)} v^j = 0$ derived from (3.2), we get

$$(3.16) \quad \delta Q^i_h = 0$$

where we have used (2.1). Since $\delta \neq 0$, therefore, the last relation yields $Q^i_h = 0$. Thus from (3.15), we have

$$(3.17) \quad W^i_{hjs} v^s + \delta_j V^i_{(h)} = 0, \quad (\delta_j = \delta_{(j)} / \delta)$$

In this way, we have

THEOREM(3.1): In order ^{that} we have (2.13), (3.17) is necessary and sufficient.

Now the equation (3.17) suggests the concrete form of the tensor q^{lm} used in the first half of the article 2. In fact $\delta_m \neq 0$ implies that there exists a suitable vector ρ^m such that

$$(3.18) \quad \delta_m \rho^m = 1$$

Then transvecting (3.17) by ρ^j and noting the above relation, we get

$$(3.19) \quad v^j_{(h)} = \delta W^j_{hj} v^s v^j$$

If, we introduce

$$(3.20) \quad q^{lm} = v^l \rho^m$$

$$\text{then } N_{lm} q^{lm} = N_{lm} v^l \rho^m = \delta_{(m)} \rho^m = \delta \delta_m \rho^m = \delta$$

i.e., from (3.17) and (2.13), we have

$$(3.21) \quad \delta = N_{lm} q^{lm}$$

Straightway. Therefore, we can take (3.20) concretely. Hence, in order to have the concrete form of q^{lm} , (3.17) should be taken as a basic condition in our theory. If this is done we are able to have (2.13) always, so $L_v W^i_{hjk} = 0$ holds good.

Thus, we have

THEOREM (3.2): If we introduce $v_{(h)}^i$ by (3.17), $L_v W_{hjk}^i = 0$ is satisfied identically.

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CHAPTER III

ON hv - RECURRENT FINSLER CONNECTION

1. INTRODUCTION

In 1934 E. Cartan ([1]) published his monograph 'Les especes de Finsler' and fixed his method to define a notion of connection in the geometry of Finsler spaces. In 1966 his method was reconsidered by M. Matsumoto ([6]) who determined uniquely Cartan's connection by assuming four elegant axioms:

- (1). The connection is metrical.
- (2). The deflection tensor field vanishes.
- (3). The torsion tensor field T vanishes.
- (4). The torsion tensor field S vanishes.

In 1969, M. Hashiguchi ([3]) replaced the condition (2) by some weaker condition and determined a Finsler connection with the given deflection tensor field. In 1975 ([4]) he also determined uniquely a Finsler connection by replacing the condition (3). In almost all these works, it has been assumed that the connection is metrical so that covariant differentiation commutes with the raising and lowering of indices.

In 1990, Prasad, Shukla and Singh ([8]) have introduced a Finsler connection with respect to which the metric tensor is h – recurrent. Such a Finsler connection has been called an h – recurrent Finsler connection. While introducing an h – recurrent Finsler connection, it has been assumed that the v – covariant derivative of metric tensor vanishes. And the torsion tensor fields T and S also vanish. The notion of hv – recurrent Finsler connection has also been studied by I.Ghinea ([2]) from another stand point.

The purpose of the present chapter is to introduce a Finsler connection, which is neither h – metrical nor v – metrical but it is recurrent with respect to both h – and v - covariant derivatives. Such a Finsler connection will be called hv – recurrent Finsler connection.

2. BASIC FORMULAE

For any Finsler connection $(F_{jk}^i, N_k^i, C_{kj}^i)$, we have five torsion tensors and three curvature tensors which are given by

(2.1) (h)h - torsion tensor: $T_{jk}^i = F_{jk}^i - F_{kj}^i$,

(2.2) (v) v - torsion tensor: $S_{jk}^i = C_{jk}^i - C_{kj}^i$,

(2.3) (h) hv - torsion tensor: C_{jk}^i = as the connection C_{jk}^i ,

(2.4) (v) h - torsion tensor: $R_{jk}^i = d_k N_j^i - d_j N_k^i$,

(2.5) (v)hv - torsion tensor: $P_{jk}^i = \partial_k N_j^i - F_{kj}^i$

(2.6) h - curvature tensor: $R_{hjk}^i = d_k F_{hj}^i - d_j F_{hk}^i + F_{hj}^m F_{mk}^i$
 $- F_{hk}^m F_{mj}^i + C_{hm}^i R_{jk}^m$,

(2.7) hv - curvature tensor: $P_{hjk}^i = \partial_k F_{hj}^i - C_{hk}^i j + C_{hm}^i P_{jk}^m$

(2.8) v - curvature tensor: $S_{hjk}^i = C_{hj}^m C_{mk}^i - C_{hk}^m C_{mj}^i$
 $+ \partial_k C_{hj}^i - \partial_j C_{hk}^i$,

The deflection tensor field D_k^i of a

Finsler connection is given by

(2.9) $D_k^i = y^j F_{jk}^i - N_k^i$.

When a Finsler metric is given, various Finsler connections are determined from the metric. The well known examples are Cartan's connection, Rund's connection and Berwald's connection. We shall use Cartan's connection which will be denoted by $(\Gamma_{jk}^i, G_k^i, C_{jk}^i)$. This connection is uniquely determined from the metric function L by the following four axioms:

- i) $g_{j|k}^i = 0, \quad g_{ij|k} = 0$
- ii) $D^i_{|k} = 0,$
- iii) $T^i_{|jk} = 0,$
- iv) $S^i_{|jk} = 0,$

and are given by

$$(2.10) \quad \Gamma^{*i}_{|jk} = 1/2 g^{ih} [d_k g_{jh} + d_j g_{kh} - d_h g_{jk}]$$

$$(2.11) \quad G^i_{|k} = \dot{\partial}_k G^i, \quad G^i = 1/2 \gamma^i_{|jk} y^j y^k,$$

$$(2.12) \quad C^i_{|jk} = 1/2 g^{ih} \dot{\partial}_h g_{jk}$$

where

$$\gamma^i_{|jk} = 1/2 g^{ih} (\partial_k g_{jh} + \partial_j g_{kh} - \partial_h g_{jk}) \text{ is the}$$

Christoffel symbol of (F^n, L) .

3. hv - RECURRENT FINSLER CONNECTIONS

Let λ_k be the component of a vector field which is positively homogeneous of degree zero in y^i and μ_k be the component of a vector field which is positively homogeneous of degree -1 in y^i . Then a Finsler connection $\{F^i_{|jk}(\lambda, \mu), N^i_{|k}(\lambda, \mu), C^i_{|jk}(\lambda, \mu)\}$ will be called hv - recurrent Finsler connection, if h - and v - covariant

derivatives of metric tensor g_{ij} with respect to this connection are recurrent, i.e., $g_{ij|k} = \lambda_k g_{ij}$ and $g_{ij||k} = \mu_k g_{ij}$. In particular if the h - covariant derivative of g_{ij} is recurrent, while v - covariant derivative of g_{ij} vanishes for a connection then that connection will be called h - recurrent Finsler connection and will be denoted by $F(\lambda, 0) = \{F^i_{jk}(\lambda), N^i_k(\lambda), C^i_{jk}(\lambda)\}$. If h - covariant derivative of g_{ij} vanishes, while v - covariant derivative of g_{ij} is recurrent for a connection then that connection will be called v - recurrent Finsler connection and will be denoted by $F(0, \mu) = \{F^i_{jk}(\mu), N^i_k(\mu), C^i_{jk}(\mu)\}$. The quantities with respect to hv - recurrent Finsler connection will be denoted by putting (λ, μ) in front of F . The quantities with respect to h - recurrent and v - recurrent Finsler connections will be denoted by $F(\lambda)$ and $F(\mu)$ respectively. The quantities without any parenthesis will correspond to the quantities with respect to the Cartan's connection $C\Gamma$. To avoid confusions we use the h - and v - covariant derivatives with respect to $C\Gamma$ by $|k$ and $||k$ while these covariant derivatives with respect to any hv - recurrent Finsler connection will be denoted by $|k$ and $||k$. To determine such an hv - recurrent Finsler connection we have the following:

THEOREM (3.1) : Given the covariant vector fields λ_k and μ_k , there exists a unique Finsler connection $F(\lambda, \mu) = \{F^i_{jk}(\lambda, \mu), N^i_k(\lambda, \mu), C^i_{jk}(\lambda, \mu)\}$ satisfying the axioms :

$$(R_1) \quad g_{ij||k} = \lambda_k g_{ij},$$

$$(R_2) \quad g_{ij||k} = \mu_k g_{ij},$$

$$(R_3) \quad \text{The deflection tensor field } D^i_k(\lambda, \mu) \text{ vanishes}$$

$$\text{i.e. } N^i_k(\lambda, \mu) = y^j F^i_{jk}(\lambda, \mu)$$

$$(R_4) \quad \text{The torsion tensor field } T^i_{jk}(\lambda, \mu) \text{ vanishes}$$

$$\text{i.e. } F^i_{jk}(\lambda, \mu) = F^i_{kj}(\lambda, \mu)$$

$$(R_5) \quad \text{The torsion tensor field } S^i_{jk} \text{ vanishes}$$

$$\text{i.e. } C^i_{jk}(\lambda, \mu) = C^i_{kj}(\lambda, \mu).$$

PROOF : From (3.1) R_1 we have

$$(3.2) \quad \partial_k g_{ij} - N^m_k(\lambda, \mu) \partial_m g_{ij} - g_{mj} F^m_{ik}(\lambda, \mu) - g_{im} F^m_{jk}(\lambda, \mu) = \lambda_k g_{ij}.$$

Applying Christoffel process to the above equation using (3.1) R_4 , (2.12) and (2.12) and (2.13) we get.

$$(3.3) \quad F_{jk}^i(\lambda, \mu) = \gamma_{jk}^i - \{C_{jm}^i N_{km}^m(\lambda, \mu) + C_{km}^i N_{mj}^m(\lambda, \mu) - g^{ni} C_{jkm} N_{mn}^m(\lambda, \mu)\} - 1/2 (\lambda_k \delta_j^i + \lambda_j \delta_k^i - \lambda^i g_{jk}) .$$

Contracting (3.3) with y^j , using the condition (3.1) R_3 and the fact that C_{jk}^i is the indicator tensor, we get

$$(3.4) \quad N_k^i(\lambda, \mu) = \gamma_{0k}^i - C_{km}^i N_{m0}^m(\lambda, \mu) - 1/2 (\lambda_k y^i + \lambda_0 \delta_k^i - \lambda^i y_k),$$

where 0 denotes the contraction with y^j i.e. $\lambda_0 = \lambda_j y^j$. Again contracting (3.4) with y^k and using (2.11), we get

$$(3.5) \quad N_0^i(\lambda, \mu) = 2 G^i - \lambda_0 y^i + 1/2 L^2 \lambda^i .$$

Substituting (3.5) into (3.4) and using the fact that the non-linear connection of Cartan's connection is

$$G_k^i = \gamma_{0k}^i - 2 C_{km}^i G^m \text{ we get}$$

$$(3.6) \quad N^i_k (\lambda, \mu) = G^i_k + T^i_k$$

where

$$(3.7) \quad T^i_k = 1/2 (\lambda^i y_k - \lambda_k y^i - \lambda_0 \delta^i_k - L^2 C^i_k)$$

$$(3.8) \quad C^i_k = \lambda^m C^i_{km}$$

Substituting (3.6) into (3.3) we get

$$(3.9) \quad F^i_{jk} (\lambda, \mu) = \Gamma^{*i}_{jk} + Q^i_{jk}$$

where Γ^{*i}_{jk} is Cartan's v -connection given

by (2.10) and

$$(3.10) \quad Q^i_{jk} = 1/2 \{ \lambda_0 C^i_{jk} + L^2 (C^i_{jm} C^m_k + C^i_{km} C^m_j -$$

$$C^m_{jk} C^i_m) - (C^i_j y_k + C^i_k y_j - C_{jk} y^i) -$$

$$(\lambda_k \delta^i_j + \lambda_j \delta^i_k - \lambda^i g_{jk})$$

From (3.1) R_2 it follows that

$$\partial_k g_{ij} - g_{mj} C^m_{ik} (\lambda, \mu) - g_{im} C^m_{jk} (\lambda, \mu) = \mu_k g_{ij}$$

Applying Christoffel process to the

above equation and using (3.1) R_5 we get

$$(3.11) \quad C^i_{jk} (\lambda, \mu) = C^i_{jk} + \sigma^i_{jk}$$

where

$$(3.12) \quad \sigma^i_{jk} = 1/2 (\mu^i g_{jk} - \mu_j \delta^i_k - \mu_k \delta^i_j)$$

From (3.9), (3.6) and (3.11) it is evident that the hv-recurrent Finsler connection $\{F^i_{jk}(\lambda, \mu), N^i_k(\lambda, \mu), C^i_{jk}(\lambda, \mu)\}$ is uniquely determined from the metric function L and from the given vector fields λ_k and μ_k .

In the following we establish the relation between (v) h-torsion tensors and (v) hv-torsion tensors corresponding to hv-recurrent Finsler connection determined in the theorem (3.1) and Cartan's connection.

From (2.4) and (3.6) we get the (v) h-torsion tensor $R^i_{jk}(\lambda, \mu)$ of hv-recurrent Finsler connection as

$$(3.13) \quad R^i_{jk}(\lambda, \mu) = R^i_{jk} + T^i_{j(k)} - T^i_{k(j)} + T^m_j \dot{\partial}_m T^i_k - T^m_k \dot{\partial}_m T^i_j,$$

where R^i_{jk} is the (v) h-torsion tensor of Cartan's connection and (k) denotes the h-covariant derivative with respect to Berwald's connection (G^i_{jk}, G^i_k, o) . The Berwald's v-connection is defined as

$$G^i_{jk} = \dot{\partial}_k G^i_j$$

and G^i_k is the same as Cartan's G^i_k .

From (2.5), (3.6) and (3.9) we get the

(v)hv-torsion tensors $P^i_{jk}(\lambda, \mu)$ of hv-recurrent Finsler connection and Cartan's connection as

$$(3.14) P^i_{jk}(\lambda, \mu) = P^i_{jk} + \dot{\partial}_k T^i_j - Q^i_{kj}$$

where P^i_{jk} is the (v)hv-torsion tensor of Cartan's connection.

From (3.6) and (3.9) we have the following

$$(3.15) \quad Q^i_{0k} = T^i_k, \quad Q^i_{jk} = Q^i_{kj}$$

Since T^i_j is positively homogeneous of degree one in y^i and $P^i_{jk} y^k = 0$, from (3.14) and (3.15) we get

$$(3.16) \quad P^i_{j0}(\lambda, \mu) = 0,$$

$$P^i_{jk}(\lambda, \mu) - P^i_{kj}(\lambda, \mu) = \dot{\partial}_k T^i_j - \dot{\partial}_j T^i_k$$

In view of (2.6), (3.6), (3.9) and (3.13) we have the following relation between h-curvature tensors of hv-recurrent Finsler connection and Cartan's connection.:

$$\begin{aligned}
(3.17) \quad R^i_{\ \ hjk}(\lambda, \mu) = & \ R^i_{\ \ hjk} + Q^i_{\ \ hjk} - Q^i_{\ \ hkj} \\
& - T^m_k \ \dot{\partial}_m \ \Gamma^{*i}_{\ \ hj} - T^m_k \ \dot{\partial}_m \ Q^i_{\ \ hj} \\
& + T^m_j \ \dot{\partial}_m \ \Gamma^{*i}_{\ \ hk} + T^m_j \ \dot{\partial}_m \ Q^i_{\ \ hk} \\
& + Q^m_{\ \ hj} \ Q^i_{\ \ mk} - Q^m_{\ \ hk} \ Q^i_{\ \ mj} + \sigma^i_{\ \ hm} \ R^m_{\ \ jk} \\
& + (C^i_{\ \ hm} + \sigma^i_{\ \ hm}) (T^m_{\ \ j(k)} - T^m_{\ \ k(j)}) \\
& + T^r_j \ \dot{\partial}_r \ T^m_k - T^r_k \ \dot{\partial}_r \ T^m_j.
\end{aligned}$$

The relation between hv-curvature tensors of hv-recurrent Finsler connection and Cartan's connection will be determined from (2.7), (3.6), (3.9) and (3.14). This relation is given by

$$\begin{aligned}
(3.18) \quad P^i_{\ \ hjk}(\lambda, \mu) = & \ P^i_{\ \ hjk} + T^m_j \ \dot{\partial}_m (C^i_{\ \ hk} + \sigma^i_{\ \ hm}) \\
& + (C^i_{\ \ hm} + \sigma^i_{\ \ hm}) \dot{\partial}_k \ T^m_j + Q^i_{\ \ hjk} - \sigma^i_{\ \ hkj} + \sigma^i_{\ \ hm} \ P^m_{\ \ jk} \\
& + Q^i_{\ \ hm} \ C^m_{\ \ jk} - \sigma^m_{\ \ hk} \ Q^i_{\ \ mj} + \sigma^i_{\ \ mk} \ Q^m_{\ \ hj}.
\end{aligned}$$

The relation between v-curvature tensors of hv-recurrent Finsler connection and Cartan's connection is obtained from (2.8) and (3.11) and this is given by

$$\begin{aligned}
(3.19) \quad S^i_{\ \ hjk}(\lambda, \mu) = & \ S^i_{\ \ hjk} + (1/2)\mu^m \{C^i_{\ \ mk} g_{hj} + C_{mhk} \delta^i_j - C^i_{\ \ mj} g_{hk} \\
& - C_{mhj} \delta^i_k\} + (1/4)\mu^2 (g_{hk} \delta^i_j - g_{hj} \delta^i_k) \\
& + (1/4) \mu^i (g_{hj} \mu_k - g_{hk} \mu_j)
\end{aligned}$$

$$+(1/4)\mu_h(\mu_j\delta^i_k - \mu_k\delta^i_j) + \dot{\partial}_k\sigma^i_{hj} - \dot{\partial}_j\sigma^i_{hk}$$

$$\text{where } \mu^2 = \mu_m \mu^m.$$

4. h-RECURRENT FINSLER CONNECTIONS

An h-recurrent Finsler connection is a particular form of hv-recurrent Finsler connection obtained by putting $\mu_k = 0$. In this section we consider a particular case of h-recurrent Finsler connection in which λ_k is the unit vector l_k and $\mu_k = 0$. Thus from (3.7), (3.10) and (3.12) we get

$$(4.1) \quad T^i_k = -(1/2)L\delta^i_k, \quad C^i_k = 0,$$

$$(4.2) \quad Q^i_{jk} = (1/2)(L C^i_{jk} - l_k\delta^i_j - l_j\delta^i_k + l^i g_{jk})$$

$$(4.3) \quad \sigma^i_{jk} = 0.$$

In view of (4.1) and (4.3) we get the following

h-recurrent Finsler connection $\{F^i_{jk}(\lambda), N^i_k(\lambda), C^i_{jk}(\lambda)\}$:

$$(4.4) \quad F^i_{jk}(\lambda) = \Gamma^*{}^i_{jk} + (1/2)(L C^i_{jk} - l_k\delta^i_j - l_j\delta^i_k + l^i g_{jk}),$$

$$(4.5) \quad N^i_k(\lambda) = G^i_k - (1/2)L\delta^i_k,$$

$$(4.6) \quad C^i_{jk}(\lambda) = C^i_{jk}$$

Since $L_{(k)} = 0$, in view of (3.13) and (4.1), we have

$$(4.7) \quad R^i_{jk}(\lambda) = R^i_{jk} + (1/4)(y_j \delta^i_k - y_k \delta^i_j)$$

Matsumoto ([7], p. 168) defined a Finsler space of scalar Curvature K which is characterized by

$$R^i_{jk} y^j = K (L^2 \delta^i_k - y_k y^i)$$

If the scalar curvature K is constant then the space (F^n, L) is said to be a Finsler space of constant curvature ([7], p.170). In view of the relation (4.7) we have the following:

THEOREM (4.1) : If the (v) h-torsion tensor $R^i_{kj}(\lambda)$ of an h-recurrent Finsler connection $F(\lambda)$ with respect to $\lambda_k = 1_k$ vanishes then (F^n, L) is of constant curvature $(-1/4)$.

Substituting (4.1) and (4.2) in (3.14) we get

$$P^i_{jk}(\lambda) = P^i_{jk} - (1/2)(L C^i_{jk} - l_j \delta^i_k + l^i g_{jk}).$$

This relation gives

$$P^i_{0k}(\lambda) = (1/2)L h^i_k, \quad P^0_{jk} = -(1/2)L h_{jk},$$

$$P^i_{jk}(\lambda) - P^i_{kj}(\lambda) = (1/2)(l_j \delta^i_k - l_k \delta^i_j),$$

which give the following :

THEOREN (4.2) : The (h) hv-torsion tensor $P^i_{jk}(\lambda)$ of an h-recurrent Finsler connection $F(\lambda)$ with respect to $\lambda_k = l_k$ never vanishes.

Substituting the values of T^i_j , Q^i_{jk} and σ^i_{jk} from (4.1), (4.2) and (4.3) in the relation (3.17), and using (2.7), (2.8), we get

$$R^i_{hjk}(\lambda) = R^i_{hjk} + (1/2)L(P^i_{hjk} - P^i_{hkj}) + (1/4)L^2 S^i_{hjk} + (1/4)(\delta^i_k g_{hj} - \delta^i_j g_{hk})$$

$$\text{Since } P^i_{hjk} - P^i_{hkj} = -\delta^i_{hjk|0}, \text{ ([7]), p.115},$$

we have the following :

THEOREM (4.3) : If the h-curvature tensor $R^i_{hjk}(\lambda)$ of an h-recurrent Finsler connection $F(\lambda)$ with respect to $\lambda_k = l_k$ vanishes and (F^n, L) is of constant curvature $(-1/4)$, then

$$S^i_{hjk|0} = (1/2)L S^i_{hjk}$$

To find the relation between the $h\nu$ - curvature tensors of an h -recurrent Finsler connection $F(\lambda)$ with respect to $\lambda_k = l_k$ and $C\Gamma$, we differentiate (4.2) v - covariantly with respect to $C\Gamma$. Then we have

$$Q^i_{hjk} = (1/2)l_k C^i_{hj} + L C^i_{hj|k} - (1/L) h_{hk} \delta^i_j - (1/L) h_{jk} \delta^i_h + (1/L) h^i_k g_{hj}.$$

Substituting (4.1), (4.2) and (4.3) in (3.18) and using (2.8) we get

$$P^i_{hjk}(\lambda) = P^i_{hjk} + (1/2)L S^i_{hjk} + (1/2)(l^i C_{jkh} - l_h C^i_{jk}) + (1/2)L(h^i_k g_{hj} - h_{hk} \delta^i_j - h_{jk} \delta^i_h).$$

Since the vertical connection $C^i_{jk}(\lambda)$ of an h -recurrent Finsler connection $F(\lambda)$ is the same as the C^i_{jk} of Cartan's connection, therefore from (2.8) it follows that

the v -curvatures of both the connections will be the same.

5. v -RECURRENT FINSLER CONNECTIONS

A v -recurrent Finsler connection

$F(\mu) = \{F^i_{jk}(\mu), N^i_k(\mu), C^i_{jk}(\mu)\}$ is a particular $h\nu$ -

recurrent Finsler connection $F(\lambda, \mu)$ obtained by putting $\lambda_k = 0$.

Thus from (3.6), (3.9) and (3.11) it follows that

$$F^i_{jk}(\mu) = \Gamma^{*i}_{jk}, \quad N^i_k(\mu) = G^i_k, \quad C^i_{jk}(\mu) = C^i_{jk} + \sigma^i_{jk}.$$

It is to be noted that our v -recurrent Finsler connection is the generalized Cartan's connection $C_\sigma \Gamma$ defined by S. Hōjō ([5]).

Putting $T^i_j = 0$ and $Q^i_{jk} = 0$ in (3.13) and (3.14) it follows that (v) h-torsion and (v) hv-torsion tensors corresponding to v - recurrent Finsler connection and Cartan's connection are identical, i.e.,

$$(5.1) \quad R^i_{jk}(\mu) = R^i_{jk}, \quad P^i_{jk}(\mu) = P^i_{jk}.$$

The relation between h-curvature tensors corresponding to v -recurrent Finsler connection and Cartan's connection is obtained from (3.17) by putting $T^i_j = 0$, $Q^i_{jk} = 0$ and the value of σ^i_{jk} from (3.12).

Thus

$$(5.2) R^i_{hjk}(\mu) = R^i_{hjk} + (1/2)(\mu^i R_{hjk} - \mu_m \delta^i_h R^m_{jk} - \mu_h R^i_{jk}).$$

Now if the Cartan's h-curvature tensor

$$R^i_{hjk} = 0,$$

then $R^i_{jk} = R^i_{hjk} y^h = 0$. Hence from (5.2) we have

$$R^i_{hjk}(\mu) = 0.$$

Conversely, if $R^i_{hjk}(\mu) = 0$ then after contracting (5.2) with y^h , we get

$$(5.3) \quad R^i_{jk} + (1/2)(\mu^i R_{0jk} - \mu_m y^i R^m_{jk} - \mu_0 R^i_{jk}) = 0.$$

Again contracting it with y^i we get,

$$(5.4) \quad R_{0jk} = (1/2) L^2 \mu_m R^m_{jk}.$$

Transvecting (5.3) with μ_i we get

$$(5.5) \quad (1 - \mu_0) R^m_{jk} \mu_m + (1/2) \mu^2 R_{0jk} = 0,$$

$$\text{where } \mu^2 = \mu_i \mu^i$$

From (5.4) and (5.5) we get $R_{0jk} = 0 = R^m_{jk} \mu_m$ provided $4(1 - \mu_0) + \mu^2 L^2 \neq 0$. Hence from (5.3) we get $R^i_{jk} = 0$ provided $\mu_0 \neq 2$. Therefore putting $R^i_{hjk}(\mu) = 0$, $R^i_{jk} = 0$ in (5.2) we get $R^i_{hjk} = 0$. Thus we get the following:

THEOREM (5.1) : If a v - recurrent Finsler connection $F(\mu)$ satisfies $4(1 - \mu_0) + \mu^2 L^2 \neq 0$, $\mu_0 \neq 2$ then the following are equivalent:

- (a) h - curvature tensor corresponding to v - recurrent Finsler connection vanishes.
- (b) h - curvature tensor of Cartan's connection vanishes.

The hv - curvature tensor of v - recurrent Finsler connection will be obtained from (3.18) and (3.12) by putting $T^i_j = 0$ and

$Q^i_{jk} = 0$. Thus

$$(5.6) P^i_{hjk}(\mu) = P^i_{hjk} - (1/2)(\mu^i_{lj}g_{hk} - \mu_{kj}\delta^i_h - \mu_{hj}\delta^i_k) +$$

$$+ \frac{1}{2} (\mu^i P_{hjk} - \mu_m P^m_{jk} \delta^i_h - \mu_h P^i_{jk}).$$

If $P^i_{hjk} = 0$ then $P^i_{jk} = P^i_{hjk} y^h = 0$. Hence from (5.6) we have $P^i_{hjk}(\mu) = 0$ provided $\mu_{ijk} = 0$. Conversely if $P^i_{hjk} = 0 = \mu_{ijk}$, then we have

$$(5.7) \quad P^i_{hjk} + (1/2)(\mu^i P_{hjk} - \mu_m P^m_{jk} \delta^i_h - \mu_h P^i_{jk}) = 0.$$

Contracting (5.7) with y^h and using the fact that $P_{hjk} y^h = 0$, we get

$$(5.8) \quad (2 - \mu_0) P^i_{jk} = \mu_m P^m_{jk} y^i.$$

Again contracting (5.8) with μ_i we get

$\mu_m P^m_{jk} = 0$ provided $\mu_0 \neq 1$. Hence (5.8) yields $P^i_{jk} = 0$ if $\mu_0 \neq 2$.

Thus (5.7) gives $P^i_{hjk} = 0$. Hence we have the following:

THEOREM (5.2): If a v - recurrent Finsler connection $F(\mu)$

satisfies $\mu_{ijk} = 0$, $\mu_0 \neq 1, 2$ then the following are equivalent:

(a) hv - curvature tensor of v- recurrent Finsler connection vanishes.

(b) hv - curvature tensor of Cartan's connection vanishes.

Since the vertical connection $C_{jk}^i(\lambda, \mu)$ of hv - recurrent Finsler connection is identical to the vertical connection $C_{jk}^i(\mu)$ of v - recurrent Finsler connection, the v - curvature tensor $S_{hjk}^i(\mu)$ of v - recurrent Finsler connection is also given by (3.19).

If the recurrence vector μ_k is such that $\mu_k = L^{-1}l_k$, then (3.19) reduces to

$$(5.9) \quad S_{hjk}^i(\mu) = S_{hjk}^i - (3/4L^2)(h_{hk}h_j^i - h_{hj}h_k^i).$$

A Finsler space of dimension $n \geq 4$ is called $S3$ - like if S_{hjk}^i is of the form ([9]):

$$L^2 S_{hjk}^i = S(h_{hj}h_k^i - h_{hk}h_j^i),$$

where S is a scalar. In this case the scalar S is a function of position alone ([10]). Therefore (5.9) gives the following:

THEOREM (5.3) : If v - curvature tensor of a v - recurrent
Finsler connection $F(\mu)$ with respect to $\mu_k = L^{-1} l_k$ vanishes
then (F^n, L) is S_3 - like. In this case $S = -3/4$.

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CHAPTER IV

ON A TRANSFORMATION OF THE FINSLER METRIC

1. INTRODUCTION

Let $F^n = (M^n, L)$ be an n - dimensional Finsler space, that is an n - dimensional differentiable manifold M^n equipped with a fundamental function $L(x, y)$. In 1971, Matsumoto [5] introduced the transformation of Finsler metric :

$$(1.1) \quad \bar{L}(x, y) = L(x, y) + X_i(x) y^i$$

$$(1.2) \quad L^{*2}(x, y) = L^2(x, y) + (X_i(x) y^i)^2$$

and obtained the relation between the imbedding class numbers of tangent Riemannian spaces to (M^n, L) , (M^n, \bar{L}) and (M^n, L^*) . Assuming $X_i(x)$ as a concurrent vector field and keeping in view the fact that a concurrent vector field is a function of coordinates only, Matsumoto ([3]) has studied the $R3$ - likeness of Finsler spaces (M^n, L) and (M^n, \bar{L}) . Singh and Prasad ([8]) and

Prasad, Singh and Singh ([7]) generalized the concept of concurrent vector field and introduced the semi – parallel and concircular vector fields which are functions of coordinates only. Assuming $X_i(x)$ as a concircular vector field, Prasad, Singh and Singh ([7]) have studied the R3 – likeness of (M^n, L) and (M^n, \bar{L}) .

If $L(x, y)$ is a metric function of Riemannian space then $\bar{L}(x, y)$ reduces to the metric function of Rander's space. Such a Finsler metric was first introduced by G. Randers ([10]) from the stand point of general theory of relativity and applied to the theory of the electron microscope by R.S. Ingarden ([1]) who first named it as Randers space. The geometrical properties of this space have been studied by ([12]), ([13]) and others. In 1978, Numata ([9]) has studied the properties of (M^n, \bar{L}) which is obtained from Minkowski space (M^n, L) by the transformation (1.1). In all these works the functions $X_i(x)$ are assumed to be functions of coordinates only.

Izumi ([2]), while studying the conformal transformation of Finsler spaces, introduced the h – vector X_i which

is v - covariantly constant with respect to Cartan's connection and satisfies the relation $LC_{ij}^h X_h = \rho h_{ij}$. Thus the h -vector X_i is not only a function of coordinates but it is also a function of directional arguments satisfying $L\dot{\partial}_j X_i = \rho h_{ij}$. Various transformations of the Finsler metric have been studied in the literature ([5]), ([6]) and ([7]). Prasad, Shukla and Singh ([6]) have obtained the relation between imbedding class numbers of tangent Riemannian spaces to (M^n, L) and (M^n, \bar{L}) , where $\bar{L}(x, y)$ is obtained from $L(x, y)$ by the transformation

$$\bar{L}(x, y) = L(x, y) + X_i(x) y^i$$

under the assumption that X_i is an h -vector in (M^n, L) .

The purpose of present chapter is to obtain the relation between v - curvature tensors with respect to Cartan's connection of the Finsler spaces (M^n, L) and (M^n, L^*) , where $L^*(x, y)$ is obtained from $L(x, y)$ by the transformation

$$L^{*2}(x, y) = L^2(x, y) + (X_i(x, y) y^i)^2,$$

where $X_i(x, y)$ is an h -vector in (M^n, L) .

2. AN h -VECTOR IN (M^n, L)

Let X_i be a vector field in the Finsler space (M^n, L) . If $X_i(x, y)$ satisfies the conditions

$$(2.1) \quad X_i|_j = 0$$

$$(2.2) \quad LC^h_{ij} X_h = \rho \delta_{ij},$$

then the vector field X_i is called an h -vector ([2]).

Here δ_j denotes the v -covariant derivative with respect to Cartan's connection $C\Gamma$, C^h_{ij} is the Cartan's C -tensor, h_{ij} is the angular metric tensor and ρ is a function given by

$$(2.3) \quad \rho = (1/n-1) L C^i X_i$$

where C^i is the torsion vector $C^i_{jk} g^{jk}$.

LEMMA (2.1) : If X_i is an h -vector then the functions ρ and $X_i^* = X_i - \rho l^i$ are independent of y .

This lemma has been proved in [2]

LEMMA (2.2) : The magnitude X of an h - vector X_i is independent of y .

This lemma has been proved in [6]

LEMMA (2.3) : For an h - vector X_i we have $S_{hijk} X^h = 0$, where S_{hijk} is v - curvature tensor of Cartan's connection $C\Gamma$.

This lemma also has been proved in [6].

Let X_i be an h - vector in the Finsler space (M^n, L) and (M^n, L^*) be another Finsler space whose fundamental metric function $L^*(x, y)$ is defined by.

$$(2.4) \quad L^{*2}(x, y) = L^2(x, y) + \mu^2(x, y),$$

where $\mu(x, y) = X_i y^i$. Since X_i is h - vector, from (2.1)

and (2.2) we get.

$$\dot{\partial}_j X_i = L^i \partial h_{ij},$$

which after using the indicatory property of h_{ij} yields

$\dot{\partial}_j \mu = X_j$. Thus differentiation of (2.4) with respect to y^i gives

$$(2.5) \quad L^* l_i^* = L l_i + \mu X_i,$$

where $l_i^* = \partial_j L^*$ is the normalized element of support in (M^n, L^*) .

The letters marked with asterisks denote the quantities of (M^n, L^*) .

Since $\partial_j l_i = L^{-1} h_{ij}$, differentiation of (2.5) with respect to y^j and application of (2.4) yield.

$$(2.6) \quad h_{ij}^* + l_i^* l_j^* = \sigma h_{ij} + l_i l_j + X_i X_j,$$

where

$$(2.7) \quad \sigma = (1 + \mu \rho / L).$$

Hence we have

$$(2.8) \quad g_{ij}^* = g_{ij} + (1 - \sigma) l_i l_j + X_i X_j$$

From (2.8), the relation between the contravariant components of the fundamental metric tensors can be derived as follows:

$$(2.9) \quad g^{*ij} = \sigma^{-1} g^{ij} - \frac{(1 - \sigma)}{L \phi} \mu (l^i X^j + l^j X^i) + \frac{(1 - \sigma)(X^2 + \sigma)}{\phi} l^i l^j + \frac{1}{\phi} X^i X^j$$

Where

$$(2.10) \phi = \frac{[\{\sigma(1-\sigma)\mu^2\} - X^2 - \sigma]}{L^2},$$

and X is the magnitude of the vector $X^i (= g^{ij} X_j)$

From the lemma (2.1) and relation (2.7) we get

$$(2.11) \quad \dot{\partial}_j \sigma = (\rho/L) u_i,$$

where

$$(2.12) \quad U_i = X_i - (\mu/L) l_i$$

Since $\dot{\partial}_k l_i = L^{-1} h_{ik}$, differentiating (2.8) with respect to y^k and using (2.4), (2.11) and (2.12) we get

$$(2.13) C^*_{ijk} = \sigma C_{ijk} + (\sigma/2L)(h_{ij} u_k + h_{ki} u_j + h_{jk} u_i).$$

From the definition of u_i , it is obvious that

$$(2.14) \quad (a) \quad u_i l^i = 0,$$

$$(b) \quad u_i X^i = X^2 - (\mu^2/L^2) = u_i u^i,$$

$$(c) \quad h_{ij} u^i = h_{ij} X^i = u_j,$$

$$(d) C_{ij}^h u_h = L^{-1} \rho h_{ij}$$

From (2.2), (2.9), (2.13) and (2.14) we get

$$\begin{aligned}
 (2.15) \quad C_{ij}^{*h} &= C_{ij}^h + (\rho/2L\sigma) (h_{ij} u^h + h_j^h u_i + h_i^h u_j) \\
 &- \left((1 - \sigma) \mu \rho / L^2 \phi \right) [\{ \sigma + (1/2)(X^2 - \mu^2/L^2) \} h_{ij} + u_i u_j] l^h \\
 &+ (\rho/L\phi) [\{ \sigma + (1/2)(X^2 - \mu^2/L^2) \} h_{ij} + u_i u_j] X^h
 \end{aligned}$$

The ν - curvature tensor S^*_{hijk} of (M^n, L^*) is defined as

$$(2.16) \quad S^*_{hijk} = C^*_{hkm} C^{*m}_{ij} - C^*_{hjm} C^{*m}_{ik}$$

From (2.13) and (2.15) we have

$$\begin{aligned}
 (2.17) \quad C^*_{hkm} C^{*m}_{ij} &= \sigma C_{hkm} C^m_{ij} + a h_{ij} h_{hk} \\
 &+ (\rho/2L) (C_{ijk} u_h + C_{ijh} u_k + C_{ihk} u_j + C_{jhk} u_i) \\
 &+ h_{hk} u_i u_j \{ \rho^2 \sigma / L^2 \phi + \rho^2 / 2 L^2 \phi \\
 &+ (\rho^2 / 2 L^2 \phi) (X^2 - \mu^2 / L^2) \} + h_{ij} u_h u_k [\rho^2 / 2 L^2 \sigma
 \end{aligned}$$

$$\begin{aligned}
 & + (\rho^2 / L^2 \phi) \{ \sigma + (1/2)(X^2 - \mu^2 / L^2) \}] \\
 & + (\rho^2 / 4 L^2 \phi) (h_{jh} u_i u_k + h_{ih} u_j u_k + h_{jk} u_i u_h + h_{ik} u_j u_h) \\
 & + (\rho^2 / L^2 \phi) u_i u_j u_h u_k
 \end{aligned}$$

where

$$\begin{aligned}
 (2.18) \quad a &= \rho^2 / L^2 + \rho^2 \sigma^2 / L^2 \phi + (\rho^2 / L^2 \phi) (X^2 - \mu^2 / L^2) \\
 & + (\rho^2 / 4 L^2 \sigma) (X^2 - \mu^2 / L^2) + (\rho^2 / 4 L^2) (X^2 - \mu^2 / L^2)^2
 \end{aligned}$$

Thus from (2.16), we get

$$(2.19) \quad S^*_{hijk} = \sigma S_{hijk} + h_{ij} d_{hk} + h_{hk} d_{ij} - h_{ik} d_{hj} - h_{hj} d_{ik},$$

where

$$(2.20) \quad d_{ij} = (1/2)a h_{ij} + b u_i u_j,$$

$$\begin{aligned}
 (2.21) \quad b &= \rho^2 \sigma / L^2 \phi + \rho^2 / 2 L^2 \sigma + (\rho^2 / 2 L^2 \phi) (X^2 - \mu^2 / L^2) \\
 & - \rho^2 / 4 L^2 \phi
 \end{aligned}$$

3. HYPERSURFACE OF (M^n, L)

Let (M^{n-1}, \underline{L}) be a hypersurface of (M^n, L) given by the equation

$$(3.1) \quad x^i = x^i(u^\alpha)$$

Let us suppose that the functions (3.1) are at least of class C^3 in u^α and the projection factor $B_{\alpha}^j = \partial x^j / \partial u^\alpha$ are such that their matrix has maximal rank $n-1$. The fundamental metric function $\underline{L}(u, v)$ of the hypersurface is given by

$$\underline{L}(u^\alpha, v^\alpha) = L [x^i(u^\alpha), B_{\alpha}^i v^\alpha],$$

where v is the element of support for the hypersurface for which

$$y^i = B_{\alpha}^i v^\alpha$$

Thus if l^α denotes the normalized vector along the element of support then

$$(3.2) \quad l^i = B_{\alpha}^i l^\alpha$$

If $g_{hj}(x, y)$ denotes the metric tensor of (M^n, L) ,
the induced metric tensor of (M^{n-1}, \underline{L}) is given by

$$(3.3) \quad g_{\alpha\beta}(u, v) = g_{hj}(x, y) B^h_\alpha B^j_\beta.$$

The inverse of (3.3) is denoted by $g^{\alpha\beta}(u, v)$ by
means of which we define the quantities.

$$(3.4) \quad B^a_i(u, v) = g^{\alpha\beta}(u, v) g_{ij}(x, y) B^j_\beta.$$

The unit normal vector $N^j(x, y)$ of (M^{n-1}, \underline{L}) is
determined by the relations

$$(3.5) \quad g_{hj}(x, y) B^h_\alpha N^j(x, y) = 0.$$

$$g_{hj}(x, y) N^h(x, y) N^j(x, y) = 1.$$

we have the following identity from (3.3), (3.4) and (3.5),

$$(3.6) \quad B^a_j B^j_\beta = \delta^\alpha_\beta, B^j_\alpha B^a_h + N^j N_h = \delta^j_h,$$

$$\text{where } N_i = g_{ij}(x, y) N^j$$

If $C_{hjk}(x, y)$ denotes the (h) hv - torsion tensor
of (M^n, L) , the induced (h) hv - torsion tensor $C_{\alpha\beta\gamma}(u, v)$ of (M^{n-1}, \underline{L})
is given by

$$(3.7) \quad C_{\alpha\beta\gamma}(u, v) = C_{hjk}(x, y) B^h_\alpha B^j_\beta B^k_\gamma,$$

from which we get

$$(3.8) \quad C^\alpha_{\beta\gamma} = B^i_\alpha C^i_{jk} B^j_\beta B^k_\gamma.$$

The relative v- covariant derivative of the projection factor B^i with respect to induced Cartan connection $IC\Gamma$ is defined as ([4])

$$(3.9) \quad B^i_{\beta\gamma} = - B^i_\alpha C^\alpha_{\beta\gamma} + C^i_{hk} B^h_\beta B^k_\gamma$$

This tensor is normal to (M^{n-1}, \underline{L}) . Therefore we may write

$$(3.10) \quad B^i_{\beta\gamma} = M_{\beta\gamma} N^i.$$

From (3.9), it is clear that $M_{\beta\gamma}$ is symmetric in β and γ and it may be written as

$$(3.11) \quad M_{\beta\gamma} = C_{ijk} N^i B^j_\beta B^k_\gamma.$$

The tangent vector space M^{n-1}_x to M^{n-1} at every point x' is regarded as $(n-1)$ - dimensional Riemannian space $(M^{n-1}_x, \underline{g}_x)$

with the Riemannian metric $\underline{g}_x = g_{\alpha\beta}(u, v) dv^\alpha dv^\beta$. The components of (h) hv - torsion tensor $C_{\beta\gamma}^\alpha$ will be the Christoffel symbols associated with \underline{g}_x . If M_x^n is the tangent vector space to M^n at $x^i (= u^\alpha)$ the $(M_x^{n-1}, \underline{g}_x)$ will be the hypersurface of (M_x^n, g_x) where $g_x = g_{ij}(x, y) dy^i dy^j$ is the Riemannian metric of M_x^n . The quantities $M_{\beta\gamma}$ given in (3.11) will be considered as the coefficients of second fundamental forms of tangent Riemannian space $(M_x^{n-1}, \underline{g}_x)$

In general the coefficients of the r th fundamental forms of $(M_x^{n-1}, \underline{g}_x)$ are defined as ([11]):

$$C_{(1)\alpha\beta} = g_{\alpha\beta}, C_{(2)\alpha\beta} = M_{\alpha\beta}, C_{(r)\alpha\beta} = C_{(r-1)\alpha\delta} M^\delta_\beta \quad (2 \leq r \leq n),$$

$$\text{where } M^\delta_\beta = g^{\alpha\delta} M_{\alpha\beta}$$

4. h - VECTOR FIELDS IN (M^{n-1}, \underline{L})

At the point of (M^{n-1}, \underline{L}) , the vector X_i may be written as

$$(4.1) \quad X_i = X_\alpha B^\alpha_i + m N_i,$$

where

$$(4.2) \quad (a) X_\alpha = X_i B^i_\alpha, (b) m = X_i N^i$$

Since $X_i|_\beta = X_{ij} B^j_\beta$, we have from (4.2(a))

and (3.10)

$$(4.3) \quad X_\alpha|_\beta = X_{ij} B^i_\alpha B^j_\beta + m M_{\alpha\beta}$$

from (3.1) and (3.6) we get

$$(4.4) \quad LC^\alpha_{\alpha\beta} X_\alpha = LC^h_{ij} X_h B^i_\beta B^j_\gamma - L m M_{\beta\gamma}$$

If X_i is a concurrent vector field in F_n then in view of (2.1), (2.2) and (3.5) the equations (4.3) and (4.4) reduce to

$$X_\alpha|_\beta = m M_{\alpha\beta}, \quad LC^\alpha_{\beta\gamma} X_\alpha = \rho h_{\beta\gamma} - L m M_{\beta\gamma}$$

In view of these relations, we have the following :

THEOREM (4.1) : If X_i is an h - vector field in (M^n, L) the vector field $X_a = X_i B^i_a$ is also an h - vector field in (M^{n-1}, \underline{L}) iff

(i) X_i is tangential to the hypersurface (M^{n-1}, \underline{L})

Or

(ii) $M_{\alpha\beta} = 0$

The hyperplanes of first, second and third kinds

are defined in [4]. In an hyperplane of third kind $M_{\alpha\beta}$ vanishes [4]. Thus :

THEOREM (4.2) : If X_i is an h-vector field in (M^n, L) then vector field $X_i B^i_a$ is also an h-vector field in a hyperplane of third kind.

In the following, we assume that X_i is tangential to (M^{n-1}, \underline{L}) so that

$$(4.5) \quad (a) \quad X_i = X_a B^a_i, \quad (b) \quad X_i = X^\alpha B^i_\alpha$$

where $X^\alpha = g^{\alpha\beta} X_\beta$

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CHAPTER V

ON THE FINSLER SPACE WITH METRIC

$$ds = (g_{ij}(y) y^i y^j)^{1/2} + X_i(x, y) y^i$$

1. INTRODUCTION

Let $F^n = (M^n, L)$ be an n-dimensional Finsler space, that is an n-dimensional differentiable manifold M^n equipped with a fundamental function $L(x, y)$. In 1974 Matsumoto ([5]) introduced the transformation of the Finsler metric:

$$(1.1) \quad L^*(x, y) = L(x, y) + b_i(x) y^i$$

and obtained the relation between the Cartan's connection coefficients of (M^n, L) and (M^n, L^*) . In 1985, [9] he also developed systematically the theory of induced Finsler connections and dealt, in particular, with the four Finsler connections namely Cartan connection, Rund Connection, Berwald connection and Hashiguchi connection. It has been assumed that the functions $b_i(x)$ in (1.1) are functions of coordinates only. If $L(x, y)$ is a metric function of Riemannian space then $L^*(x, y)$ reduces to the metric function of Randerspace. Such a Finsler metric was first introduced by G. Randers ([7]) from the view point of general theory of relativity and applied to the theory of electron microscope by

R.S. Ingarden ([1]), who first named it a Randers space. In the papers ([4]), ([5]) and ([8]) this space has been studied from a geometrical view point. In 1978, Numata ([6]) has obtained the torsion tensors R_{hjk} and P_{hjk} of (M^n, L^*) which is obtained from Minkowskian space (M^n, L) by the transformation (1.1). In all these works the functions b_i are assumed to be a function of coordinates only.

Izumi [2], while studying the conformal transformation of Finsler spaces, introduced the h-vector X_i which is v-covariantly constant with respect to Cartan's connection $C\Gamma$ and satisfies the relation $LC^h_{ij} X_h = \rho h_{ij}$. Thus the h-vector X_i is not only a function of coordinates but is also a function of directional arguments satisfying $L \partial_j X_i = \rho h_{ij}$. In the second section of this chapter we shall find out the relation between Cartan's connection $C\Gamma$ of (M^n, L) and (M^n, L^*) where $L^*(x, y)$ is obtained from $L(x, y)$ by the transformation:

$$(1.2) \quad L^*(x, y) = L(x, y) + X_i(x, y) y^i$$

under the assumption that $X_i(x, y)$ is an h-vector in (M^n, L) .

In the third section of this chapter, we shall introduce a Finsler space with a metric $ds = \mu + \beta$, where $\mu = [g_{ij}(dx^i dx^j)]^{1/2}$ is a Minkowskian metric and $\beta = X_i(x, y) y^i$, where X_i is an h-vector in (M^n, L) . We shall find out the torsion tensor R^*_{hjk} of F^{*n} and consider the case that this space is of scalar curvature. The fourth section is devoted to find the torsion tensor P^*_{hjk} and to consider the case that this space is a Landsberg space.

2. CARTAN'S CONNECTION OF THE SPACE F^{*n}

Let X_i be a vector field in the Finsler space (M^n, L) . If X_i satisfies the conditions

$$(2.1) \quad (i) X_i|_j = 0, \quad (ii) LC^h_{ij} = \rho h_{ij}$$

then the vector field X_i is called an h-vector ([2]). Here in the above equation (2.1) $|_j$ denotes the v-covariant derivative with respect to Cartan's connection $C\Gamma$, C^h_{ij} is the Cartan's C-tensor, h_{ij} is the angular metric tensor and ρ is a function given by

$$(2.2) \quad \rho = \{1/(n-1)\} L C^i X_i, \quad C^i = C_{jk}^i g^{jk}$$

from (2.1) we get

$$(2.3) \quad \dot{\partial}_j X^i = L^{-1} \rho h_{ij}$$

If we denote $X_i y^i$ as β then indicatory property of h_{ij} yields $\dot{\partial}_j \beta = X_j$

Thus differentiation of (1.2) with respect to y^i gives

$$(2.4) \quad L_i^* = L_i + X_i$$

Throughout the chapter, we shall use the notations

$$L_i = \dot{\partial}_i L, \quad L_{ij} = \dot{\partial}_j \dot{\partial}_i L \text{ etc.}$$

The quantities and operations referring to F^n are marked with asterisks. Thus from (1.2) we get

$$(2.4) \quad (a) l_i^* = l_i + X_i$$

where l_i is the normalized element of support. Again from (2.3) and (2.4) we get

$$(2.5) \quad L_{ij}^* = (1+\rho) L_{ij}$$

If $g_{ij} = 1/2 \partial_j \partial_i$, L^2 denotes the metric tensor of F^n then the angular metric tensor h_{ij} of F^n is given by

$$(2.6) \quad h_{ij} = g_{ij} - l_i l_j = L^{-1} L_{ij}$$

Thus (2.5) may be rewritten as

$$(2.5)(a) \quad h_{ij}^* = \tau(1+\rho) h_{ij},$$

where $\tau = L^{-1} L^*$

By virtue of (2.6), (2.5) (a) and (2.4)(a) the relation between fundamental tensors is given by

$$(2.7) \quad g_{ij}^* = \tau(1+\rho) g_{ij} + \{1-\tau(1+\rho)\} l_i l_j + (l_i X_j + l_j X_i) + X_{\lambda} X_j$$

from (2.7) the relation between contravariant components of the fundamental tensors will be derived as follows:

$$(2.8) \quad g^{*ij} = [\tau(1+\rho)]^{-1} g^{ij} - \tau^3 (1+\rho)^{-1} \{1-X^2 - \tau(1+\rho)\} l^i l^j - \tau^2 (1+\rho)^{-1} (l^i X^j + l^j X^i),$$

Where X is the magnitude of the vector $X^i = g^{ij} x_j$.

By virtue of lemma (2.1) of chapter IV and (2.5) it follows that all the successive derivatives of L_{ij}^* with respect to y^k are

proportional to the corresponding successive derivatives of L_{ij} with factor of proportionality $(1+\rho)$, i.e.

$$(2.9) \quad (a) \quad L_{ijk}^* = (1+\rho)L_{ijk}, \quad (b) \quad L_{ijkh}^* = (1+\rho)L_{ijkh} \text{ and so on.}$$

From the equations

$$(2.10) \quad L_{ijk} = 2L^{-1}C_{ijk} - L^{-2}(h_{ij}l_k + h_{jk}l_i + h_{ki}l_j), \quad C_{ijk} = 1/2\partial_k g_{ij},$$

(2.4)(a), (2.5)(a) and (2.9)(a), we obtain the following relation between C_{ijk} and C_{ijk}^* :

$$(2.11) \quad C_{ijk}^* = \tau(1+\rho)C_{ijk} + \{ (1+\rho)/2L \} (h_{ij}u_k + h_{jk}u_i + h_{ki}u_j),$$

where we put

$$(2.12) \quad u_i = X_i - \beta L^{-1}l_i$$

from (2.1), (2.8) and (2.11) we get

$$(2.13) \quad C_{ij}^h = C_{ij}^h + (2L^*)^{-1} (h_{ij}u^h + h_j^h u_i + h_i^h u_j) \\ - L^{*-1} \{ \rho + (2L^*)^{-1} L(X^2 - \beta^2 L^{-2}) \} h_{ij} \\ + LL^{*-1} u_i u_j l^h$$

Now we shall be concerned with Cartan's connection of F^n and F^{*n} . This connection is denoted by $C\Gamma = (F^i_{jk}, N^i_k, C^i_{jk})$.

Here $N^i_k = N^i_{0k}$ ($= y^j F^i_{jk}$) and $C^h_{ij} = g^{hk} C_{ikj}$

Since for a Cartan's connection

$$0 = D_{ijk} = \partial_k L_{ij} - L_{ijr} N^r_k - L_{rj} F^r_{ik} - L_{ir} F^r_{jk},$$

We obtain

$$(2.14) \quad \partial_k L_{ij} = L_{ijr} N^r_k + L_{rj} F^r_{ik} + L_{ir} F^r_{jk},$$

Differentiation of (2.5) leads to

$$(2.15) \quad \partial_k L^*_{ij} = (1+\rho) \partial_k L_{ij} + \rho_k L_{ij},$$

where $\rho_k = \partial_k \rho$. If we put

$$(2.16) \quad D^i_{jk} = F^*_{jk} - F^i_{jk}$$

then the difference D^i_{jk} is obviously a tensor of (1,2) type. In view of (2.14) the equation (2.15) is written in the tensorial form

$$(2.17) \quad (1+\rho) (L_{ijr} D^r_{0k} + L_{rj} D^r_{ik} + L_{ir} D^r_{jk}) = \rho_k L_{ij}.$$

In order to find the difference D^i_{jk} we construct supplementary equations to (2.17). From (2.4) we obtain

$$(2.18) \quad \partial_j L^*_i = \partial_j L_i + \partial_j X_i$$

From $L_{ij} = 0$, the equation (2.18) is written in the form

$$L_{ir}^* N_j^r + L_r^* F_{ij}^r = (1+\rho) L_{ir} N_j^r + (L_r + X_r) F_{ij}^r + X_{ij}$$

By means of (2.4), (2.5) and (2.16) this equation may be written in the tensorial form

$$(2.18)(a) \quad (1+\rho) L_{ir} D_{0j}^r + (L_r + X_r) D_{ij}^r = X_{ij}$$

To find the difference tensor D_{jk}^i we have the following ([4]):

LEMMA (2.1) The system of algebraic equations

$$(i) L_{ir} A^r = B_i, \quad (ii) (L_r + X_r) A^r = B$$

has a unique solution A^r for given B and B_i such that $B_i l^i = 0$. The solution is given by $A^i = LB^i + \tau^{-1} (B - L B_\beta) l^i$, where the subscript β denotes the contraction by X^i .

Now we establish the following :

THEOREM (2.1): the Cartan's connection of F^{*n} is completely determined by the equations (2.17) and (2.18) (a) in terms of the one of F^n .

PROOF: It is obvious that (2.18) (a) is equivalent to the two equations

$$(2.19) \quad (1+\rho) (L_{ir} D_{0j}^r + L_{jr} D_{0i}^r) + 2 (L_r + X_r) D_{ij}^r = 2 E_{ij}$$

$$(2.20) \quad (1+\rho) (L_{ir} D^r_{0j} - L_{jr} D^r_{0i}) = 2F_{ij}$$

where we put

$$(2.21) \quad 2E_{ij} + X_{ilj} + X_{jik}, \quad 2F_{ij} = X_{ij} - X_{ji}$$

On the other hand (2.17) is equivalent to

$$(2.22) \quad 2(1+\rho) L_{jr} D^r_{ik} + (1+\rho) (L_{ijr} D^r_{0k} + L_{jkr} D^r_{0i} - L_{kir} D^r_{0j}) \\ = \rho_k L_{ij} + \rho_i L_{jk} - \rho_j L_{ki}$$

Contracting (2.19) with y^j , we get

$$(2.23) \quad (1+\rho) L_{ir} D^r_{00} = 2(l_r + X_r) D^r_{0i} = 2E_{i0}$$

Similarly from (2.20) and (2.22) we obtain

$$(2.24) \quad (1+\rho) L_{ir} D^r_{00} = 2F_{i0}$$

$$(2.25) \quad (1+\rho) (L_{ir} D^r_{0j} + L_{jr} D^r_{0i} + L_{ijr} D^r_{00}) = \rho_0 L_{ij}$$

Contraction of (2.23) with y^i gives

$$(2.26) \quad (l_r + X_r) D^r_{00} = E_{00}$$

Now first consider (2.24) and (2.26) and apply lemma (2.1) to obtain

$$(2.27) \quad D^i_{00} = (1+\rho)^{-1} 2 L F^i_0 + \tau^{-1} (E_{00} - 2L (1+\rho)^{-1} F_{\beta 0}) l^i$$

where we put $F^i_0 = g^{ij} F_{j0}$

Secondly we add (2.20) and (2.25) to obtain

$$(2.28) \quad L_{ir} D^r_{0j} = G_{ij}$$

where we put

$$(2.29) \quad G_{ij} = (2(1+\rho))^{-1} (2F_{ij} + \rho_0 L_{ij} - (1+\rho) L_{ijr} D^r_{00})$$

The equation (2.23) is written in the form

$$(2.23)(a) \quad (l_r + X_r) D^r_{0j} = G_j$$

where we put

$$(2.30) \quad G_j = E_{j0} - 2^{-1} (1+\rho) L_{jr} D^r_{00}$$

Substitution from (2.27) in (2.29) yields

$$(2.29)(a) \quad G_{ij} = (1+\rho)^{-1} [F_{ij} - LL_{ijr} F^r_0 + L_{ij} \{(1+\rho) E_{00} - 2LF_{\beta 0} + L^* \rho_0 \} (2L^*)^{-1}]$$

By virtue of (2.24), G_j are written as

$$(2.30)(a) \quad G_j = E_{j0} - F_{j0}$$

Thus we have obtained the system of equations

(2.28) and (2.23)(a). Applying Lemma (2.1) to these equations we obtain

$$(2.31) \quad D^i_{0j} = LG^i_j + \tau^{-1} (G_j - LG_{\beta j}) l^i,$$

where we put $G^i_j = g^{ir} G_{rj}$

Finally from (2.19) and (2.22), we obtain

$$(A) \quad L_{ir} D^r_{jk} = H_{ijk}, (l_r + X_r) D^r_{jk} = H_{jk}$$

where we put

$$(2.32) \quad H_{ijk} = \{2(1+\rho)\}^{-1} (\rho_k L_{ij} + \rho_j L_{ik} - \rho_i L_{kj})$$

$$- (1/2)(L_{ijr} D^r_{0k} + L_{ikr} D^r_{0j} - L_{kjr} D^r_{0i}),$$

$$H_{jk} = E_{jk} - \{(1+\rho)/2\} (L_{jr} D^r_{0k} + L_{kr} D^r_{0j})$$

Now applying Lemma (2.1) to (A), we get

$$(2.33) \quad D^i_{jk} = L H^i_{jk} + \tau^{-1} (H_{jk} - L H_{\beta jk}) l^i$$

where we put $H^i_{jk} = g^{hi} H_{hjk}$. By virtue of

(2.31), H_{ijk} and H_{jk} are written in terms of known quantities:

$$(2.34) \quad H_{ijk} = (1/2)L(L_{kjr} G^r_i - L_{ijr} G^r_k - L_{ikr} G^r_j)$$

$$+ L_{ij} A_k + L_{ik} A_j - L_{jk} A_i,$$

$$(2.35) \quad H_{jk} = E_{jk} - (1+\rho) L^{-2^{-1}} (L_{jr} G^r_k + L_{kr} G^r_j)$$

where

$$(2.36) \quad A_i = \{2(1+\rho)\}^{-1} \rho_i + (2\tau)^{-1} (G_i - L G_{\beta i})$$

3. THE (v) h - TORSION TENSOR R^*_{hjk} OF F^n

Let F^n be a locally Minkowskian space, where fundamental function is expressed by $L(y) = \{g_{ij}(y) y^i y^j\}^{1/2}$ in terms

of an adaptable coordinate system (x^i). For a given h-vector X_i in F^n , we obtain another Finsler space F^{*n} with the fundamental function

$$(3.1) \quad L^*(x, y) = L(y) + \beta(x, y),$$

where $\beta(x, y) = X_i(x, y) y^i$. With reference to the adaptable coordinate system (x^i) the connection parameters ($F_{jk}^i, N_j^i, C_{jk}^i$) of the Cartan connection of F^n are given by

$$(3.2) \quad F_{jk}^i = 0, N_j^i = F_{oj}^i = 0, C_{jk}^i = g^{ir} C_{rjk} = (1/2) g^{ir} \partial_k g_{rj}.$$

Thus the h-covariant differentiation X_{ij} of a covariant vector field X_i may be written as $\partial_j X_i$ and the v-covariant differentiation of X_i as $X_{ij} = \partial_j X_i - X_r C_{rj}^i$. In view of (2.16), (2.13) and (3.2) the connection parameter N^{*i}_j of F^{*n} may be written as

$$(3.3) \quad N^{*i}_j = L G_j^i + \tau^{-1} (G_j - L G_{\beta j}) l^i.$$

In view of (2.10) and (2.29) (a) the value of G_{ij} may be written as

$$(3.4) \quad G_{ij} = (1+\rho)^{-1} [A_{ij} + L^{-1} (F_{j\alpha} l_i + F_{i\alpha} l_j) + G h_{ij}]$$

where

$$(3.5) \quad G = (2LL^*)^{-1} \{(1+\rho) E_{00} - 2L F_{\beta 0} + L^* \rho_0\}$$

and

$$(3.6) \quad A_{ij} = F_{ij} - 2C_{ijr} F^r_0$$

The (v) h-torsion tensor R^*_{hjk} of (M^n, L^*) is defined by

$$(3.7) \quad R^*_{hjk} = g^*_{hi} R^*_{jk} = h^*_{hi} R^*_{jk}$$

$$= \mathcal{L}_{(j,k)} \{ h^*_{hi} (\partial_k N^*_{j} - N^*_{r} \partial_r N^*_{j}) \}$$

In view of (2.5) (a) and (2.6), we have

$$(3.8) \quad R^*_{hjk} = \mathcal{L}_{(j,k)} \{ (1+\rho) L^* L_{hi} (\partial_k N^*_{j} - N^*_{r} \partial_r N^*_{j}) \}$$

By virtue of (3.2) and (2.16) the equation (2.28) may be written as $L_{hi} N^*_{j} = G_{hj}$ from which we get

$$L_{hi} \partial_k N^*_{j} = G_{hjk}$$

and

$$\mathcal{L}_{(j,k)} \{ L_{hi} N^*_{r} \partial_r N^*_{j} \} = \mathcal{L}_{(j,k)} \{ L G^r_k \partial_r G_{hj} \}$$

Thus (3.8) may be written as

$$(3.9) \quad R^*_{hjk} = [(1+\rho) \mathcal{L}_{(j,k)} \{ L^* (G_{hjk} - L G^r_k \partial_r G_{hj}) \}]$$

By virtue of (3.4) we have

$$(3.10) \quad G_{hjk} = (1+\rho)^{-1} [A_{hjk} + L^{-1} (l_h F_{jok} + l_j F_{hok}) + G_{jk} h_{hj}]$$

$$\begin{aligned}
 & -(1+\rho)^{-2} \rho_k \{ A_{hj} + L^{-1} (l_h F_{jo} + l_j F_{ho}) + G h_{hj} \}, \\
 (3.11) \quad & \dot{\partial}_r G_{hj} = (1+\rho)^{-1} [-2(F_{uo} \dot{\partial}_r C_{hj}^u + C_{hj}^u F_{ur} \\
 & + \dot{\partial}_r G h_{hj} + \{ G - \rho_0 (2L)^{-1} \} \{ 2C_{hjr} - L^{-1} (l_h h_{jr} + l_j h_{hr}) \} \\
 & + L^{-2} \{ (h_{hr} - l_h l_r) F_{jo} + (h_{jr} - l_j l_r) F h_0 \} \\
 & + L^{-1} \{ l_h F_{jr} + l_j F_{hr} + 2^{-1} (\rho_j h_{hr} - \rho_h h_{jr}) \}]
 \end{aligned}$$

From (3.4) and (3.11) we get

$$\begin{aligned}
 (3.12) \quad & (1+\rho)^2 \{ \zeta_{(j,k)} \{ G^r_k \dot{\partial}_r G_{hj} \} \\
 & = \zeta_{(j,k)} \{ -[A^r_j \dot{\partial}_r G \\
 & + G \dot{\partial}_j G + L^{-1} l_j (F^r_0 \dot{\partial}_r G + G^2) - L^{-2} G (F_{jo} - 2^{-1} \rho_0 \\
 & l_j + 2^{-1} L \rho_j)] h_{hk} + 2A^r_j (F_{so} \dot{\partial}_r C^s_{hk} + C^s_{hk} F_{sr} + (2L)^{-1} \rho_0 C_{hkr}) + 2G F_{so} \\
 & (\dot{\partial}_j C^s_{hk} + 2C^s_{jr} C^r_{hk}) \\
 & - L^{-2} (A_{hj} F_{ko} - F_{ho} F_{jk} - F^r_0 F_{jr} l_h l_k) \\
 & - L^{-1} [A^r_j F_{hr} l_k + 2F^r_0 (F_{so} \dot{\partial}_r C^s_{hj} + C^s_{hj} F_{sr}) l_k + 2F^r_0 C^s_{rj} F_{sk} l_j] \\
 & - L^{-2} \rho_0 C_{hjr} F^r_0 l_k + 2^{-1} L^{-2} \rho_0 (l_h A_{jk} + l_j A_{hk} + L^{-1} l_h l_j A_{k0}) \\
 & + 2^{-1} L^{-1} (\rho_j A_{hk} - \rho_h A_{jk}) + 2^{-1} L^{-2} \rho_j (l_h A_{k0} + l_k F_{h0}) \}
 \end{aligned}$$

Substituting from (3.10) and (3.12) in (3.9),

we obtain

THEOREM(3.1) : The (v)h- torsion tensor R^*_{hjk} of the space F^{*n} is written in the form

$$(3.13) \quad R^*_{hjk} = (1+\rho)^{-1} \left\{ L^* L G'_j h_{hk} + L^2 K_{hjk} + L(l_h K_{jk} + l_j K_{hk}) - l_h l_j K_{0k} \right\}$$

where

$$G'_j = A^r_j \dot{\partial}_r G + G \dot{\partial}_j G - L^{-1} \{ G_{jj} (1+\rho) - (F^r_0 \dot{\partial}_r G + G^2) l_j \}$$

$$- L^{-2} G F_{j0} + 2^{-1} L^{-2} (L \rho_j - \rho_0 l_j),$$

$$K_{hjk} = \tau [L^{-1} (1+\rho) A_{hjk} - 2 A^r_j (F_{s0} \dot{\partial}_r C^s_{hk} + F_{sr}) - 2 G F_{s0} (\dot{\partial}_j C^s_{hk} + 2 C^s_{jr} C^r_{hk}) + L^{-2} (A_{hj} F_{k0} - F_{h0} F_{jk}) + \rho_0 L^{-1} C_{hjr} A^r_k + (2L)^{-1} (\rho_j A_{hk} + \rho_h A_{jk})]$$

$$K_{jk} = K_{j0k} - \tau \{ A^i_k F_{ji} + 2 G C^s_{jk} F_{s0} + L^{-1} (2 F_{j0} F_{ko} + \rho_0 A_{jk} + \rho_0 C_{jki} F_{i0}) + (2L)^{-1} (\rho_k F_{j0} + \rho_j F_{ko}) \}$$

Now we shall be concerned with the contracted tensor R^*_{i0j} of R^*_{ikj} . The space F^{*n} is of scalar curvature R^* if the equation $R^*_{i0j} = R^* L^{-2} h^*_{ij}$ holds good ([3]). If the scalar $R^* =$ constant, then F^{*n} is said to be of constant curvature.

From (3.13) the contracted (v) h-torion tensor

R^*_{i0j} of F^{*n} is given by

$$(3.14) \quad R^*_{i0j} = (1+\rho)^{-1} \{ L^* L G'_0 h_{ij} + L^2 W_{ij} - L (l_i W_{j0} + l_j W_{i0}) + W_{00} l_i l_j \},$$

where we put

$$W_{ij} = K_{i0j} - K_{ij0} + K_{ij}$$

and W_{ij} is symmetric in the indices i and j . It is to be

noted that $R^*_{i0j} = R^* L^{*2} h_{ij}^*$ is rewritten as $R^*_{i0j} = \tau(1+\rho) R^* L^{*2} h_{ij}$.

Therefore we obtain easily from (3.14) the following:

THEOREM (3.2) : Let F^{*n} be a Finsler space with a metric

$$L^* = L + \beta \quad \text{Where } L = \{g_{ij}(y) y^i y^j\}^{1/2}, \quad \beta = X_i(x, y) y^i \text{ and } X_i$$

is an h-vector in F^n . If F^{*n} is of scalar curvature R^* then the

matrix $\|\lambda h_{ij} - W_{ij}\|$ is of rank less than three, where we put

$$\lambda = \tau \{(1+\rho)^2 \tau^2 R^* - G'_0\} .$$

Now we consider the case $F_{ij} = 0$. In this case

$A_{ij} = 0$, $K_{hjk} = 0$, $K_{hj} = 0$ and $W_{ij} = 0$ hold good. Therefore the tensor

$$R^*_{i0j} = \{1/(1+\rho)\} L^* L G'_0 h_{ij} . \quad \text{Consequently, we}$$

obtain

THEOREM (3.3) : Let F^{*n} be an above mentioned Finsler space If

the condition $F_{ij} = 0$ is satisfied, then F^{*n} is of scalar curvature

$$R^* = \{(1+\rho)\tau\}^2 G'_0 .$$

Though the concept of a Finsler space of scalar

curvature was introduced by L. Berwald in 1947, we have no

concrete example of non-Minkowski space of scalar curvature.

Moreover we can show the following:

THEOREM (3.4) : In Theorem (3.3) if the scalar curvature R^* is constant, then $R^* = 0$ and the space F^{*n} is a locally Minkowskian space.

PROOF : From (2.3) and $F_{ij} = 0$, we get

$$(3.15) \quad 2\partial_r F_{ij} = L^{-1}(\rho_j h_{ir} - \rho_i h_{jr}) = 0.$$

which after contraction with y^j gives $\rho_0 = 0$. Thus contracting (3.15) with g^{jr} we get $\rho_j = 0$. Therefore the scalar R^* is written in the form

$$(3.16) \quad R^* = \{ (1+\rho)\tau \}^{-2} \{ G^2 - L^{-1} (1+\rho) G_{|0} \} .$$

It follows from (3.16) and $G = \{(1+\rho)/2LL^*\}E_{00}$ that the condition $R^* = C$ ($=$ Constant) is written in the form

$$(3.17) \quad [2\beta E_{00|0} - 3E^2_{00} + 4(L^4 + 6L^2\beta^2 + \beta^4)C] \\ + 2L[E_{00|0} + 8\beta(L^2 + \beta^2)C] = 0.$$

The term in the first (resp. second) bracket of the left hand side of (3.17) is a polynomial of the fourth (resp. third) order with respect to y^i . Therefore (3.17) is equivalent to

$$(3.18) \quad 2\beta E_{00|0} - 3E_{00}^2 + 4(L^4 + 6L^2\beta^2 + \beta^4)C = 0,$$

$$(3.19) \quad E_{00|0} + 8(L^2 + \beta^2)C = 0.$$

From (3.18) and (3.19) we obtain

$$(3.20) \quad 3E_{00}^2 = 4C(L^2 - \beta^2)(L^2 + 3\beta^2).$$

If C doesn't vanish then in view of $F_{ij}=0$ and $\beta_{|0}=E_{00}$, the h-covariant differentiation of (3.20) yields

$$(3.21) \quad 3E_{00|0} = 8\beta C(L^2 - 3\beta^2).$$

Elimination of $E_{00|0}$ from (3.19) and (3.21) gives

$L^2\beta C = 0$ from which we get $\beta = 0$ as $L^2 C \neq 0$. Since $\partial_j \beta = X_j$, therefore $\beta = 0$ implies $X_j = 0$, $E_{ij} = 0$. Hence (3.20) gives $C = 0$ as $L^2 - \beta^2 \neq 0$, $L^2 + 3\beta^2 \neq 0$. This contradicts our assumption $C \neq 0$. Hence the scalar $R^* = C = 0$ and from (3.20) we get $E_{00} = 0$. Since the assumption $F_{ij} = 0$ implies $\rho_i = 0$, therefore $E_{00} = 0$ implies that $F_{ij} = 0$ so that $X_{ij} = \partial_j X_i = 0$. Thus X_i does not contain x^i . Thus F^{*n} is locally Minkowskian space.

4. THE (v)hv-TORSION TENSOR P^*_{hjk} OF F^{*n}

We shall continue to be concerned with the above mentioned Finsler space F^{*n} . The (v)hv - torsion tensor P^*_{hjk} of F^{*n} is defined as

$$(4.1) \quad P^*_{hjk} = C^*_{hjk|0} = y^r \partial_r C^*_{hjk} - (\dot{\partial}_r C^*_{hjk}) N^r_0 - \sum_{(h,j,k)} \{ C_{hjr} F^*_{rk0} \}$$

Where and in the following the symbol $\sum_{(h,j,k)}$ denotes the cyclic permutation of h,j,k and summation.

In view of (2.11) and $P_{hjk} = C_{hjk|0} = 0$, we obtain

$$(4.2) \quad y^r \partial_r C^*_{hjk} = C^*_{hjk|0} = 2(L^*G + F_{\beta 0}) C_{hjk} + \sum_{(h,j,k)} \{ (2L)^{-1} (\rho_0 u_k + (1+\rho) (X_{k|0} - L^{-1} G_0 l_k) h_{hj} \}$$

$$(4.3) \quad \dot{\partial}_r C^*_{hjk} = \tau (1+\rho) \dot{\partial}_r C_{hjk} + L^{-1} (1+\rho) C_{hjk} u_r + \sum_{(h,j,k)} \{ (1+\rho) L^{-1} C_{hjr} u_k - (2L^2)^{-1} (1+\rho) h_{hj} (n_{kr} + (\rho - \beta L^{-1}) h_{kr}) + (2L^2)^{-1} (1+\rho) h_{hr} n_{jk} \}$$

where we put $n_{ij} = l_i u_j + l_j u_i$. Therefore (3.3), (3.4) and (4.3) lead us

to

$$(4.4) \quad \dot{\partial}_r C^*_{hjk} N^r_0 = 2 L^* \dot{\partial}_r C_{hjk} F^r_0 - (2L^* G - \tau \rho_0 - 2F_{\beta 0}) C_{hjk}$$

$$+ \sum_{(h,j,k)} \{ 2F_{r0} C^r_{hj} u_k - L^{-1} F_{ho} n_{jk} - h_{hj} (L^{-1} F_{\beta 0} l_k - L^{-1} (\rho - \beta L^{-1}) F_{ko} + (G - (2L)^{-1} \rho_0) u_k) \}.$$

By virtue of (3.3), (3.4) and (2.11) we have

$$(4.5) \quad \sum_{(h,j,k)} \{ C^*_{hjr} F^{*r}_{ko} \} = 3L^* G C_{hjk} + \sum_{(h,j,k)} \{ L^* C^r_{hj} (A^r_k + L^{-1} F_{r0} l_k) - 2 C^r_{hj} F_{r0} u_k + L^{-1} F_{ho} n_{jk} + 2^{-1} h_{ij} (A_{\beta k} + L^{-1} F_{\beta 0} l_k + L^{-2} \beta F_{ko} + 3G u_k) \} .$$

Substituting from (4.2), (4.4) and (4.5) in (4.1) we obtain

THEOREM (4.1) : The (v) hv - torsion tensor P^*_{hjk} of the space F^{*n}

is written in the form

$$P^*_{hjk} = -2\tau T_{hjkr} F^r_0 + (L^* G - \tau \rho_0) C_{hjk} + \sum_{(h,j,k)} \{ \tau C^r_{hj} (F_{r0} l_k + L F_{kr}) + h_{hj} P_k \}$$

where we put

$$T_{hjkr} = L C_{hjk|r} + C_{hjk} l_r + \sum_{(h,j,k)} \{ C_{rjk} l_h \}$$

$$2 P_k = -A_{\beta k} + L^{-1} [(1+\rho) E_{k0} + (\tau-\rho) F_{k0} - (F_{\beta 0} + 2L^* G - \tau \rho_0) l_k]$$

$$- G u_k.$$

If the condition $F_{ij} = 0$ is satisfied then the (v) hv - torsion tensor P^*_{hjk} of F^{*n} is given by

$$(4.6) \quad P^*_{hjk} = (L^* G - \tau \rho_0) C_{hjk} + \sum_{(h,j,k)} \{ h_{hj} P_k \}$$

Where

$$G = (2L L^*)^{-1} [(1+\rho) E_{00} + L^* \rho_0]$$

$$2 P_k = L^{-1} [(1+\rho) F_{k0} - (2L^* G - \tau \rho_0) L_k] - G u_k$$

Now we shall treat a Landsberg space F^{*n} . Such a space is by definition a Finsler space with the (v) hv - torsion tensor $P_{hjk}^* = C_{hjk|0}^* = 0$. On the other hand a Finsler space F^{*n} with $C_{hijkl}^* = 0$ is called a Berwald space (or an affinely connected space).

THEOREM (4.2) : Let F^{*n} ($n \geq 3$) be a Finsler space with a metric L^* $= L + \beta$, where $L = \{g_{ij}(dx) dx^i dx^j\}^{1/2}$, $\beta = X_i(x, y) y^i$ and X_i is an h - vector in (M^n, L) . In the case $F_{ij} = 0$, if F^{*n} is a Landsberg space then F^{*n} is reduced to a Berwald space.

PROOF: It is easy to see that the condition $(L^* G - \tau \rho_0) = 0$ means $E_{00} = 0$, i.e., $E_{ij} = 0$. From $E_{ij} = F_{ij} = 0$ we have $X_i = \text{constant}$, so that F^{*n} reduces to a locally Minkowskian space. In the case $L^* G - \tau \rho_0 \neq 0$, by virtue of (4.6), the equation $P_{hjk}^* = 0$ is equivalent to

$$C_{hjk} = -(L^* G - \tau \rho_0)^{-1} \sum_{(h,j,k)} \{h_{hj} P_k\},$$

that is, F^n is C - reducible. By virtue of Theorem 1 of ([10]), the space F^{*n} ($n \geq 3$) turns out to be a Berwald space. Consequently the proof of Theorem (4.2) is complete.

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